Contents

Chapter 1

SET THEORY
(INCLUDING RELATION AND FUNCTION)

1. MEANING, DEFINITION AND CHARACTERISTICS

Meaning

Set Theory is the branch of Mathematical Logic that studies sets. **A set is a Collection of well defined objects.** In common parlance, a set means the whole or entirety of a thing made or found in different parts. For example, a set of Mahabharat, a set of Ramayan etc. But in Mathematics, by set we mean a collection of well defined objects whether available or not but, expressed in the form of a notation or a diagram and capable of algebraic treatment under certain theoretical concepts.

The examples of such sets are :

(i) The Collection of all the districts of Odisha is a set.

(ii) The Collection of past Presidents of Indian Union is a Set.

(iii) The Collection of first Five prime natural numbers is a set.

(iv) The vowels in English alphabet. This set contains by elements namely [a, e, i, o, u] is a set.

(v) Cricketers in the world who were out at zero in a test match, is a set.

It is to be noted that Collection of good cricket players of India is not a set. Since, the term ‘good player’ is a vague one and is not well defined.

The theory of set is a recent development in the horizon of mathematics i.e., after 1870. It was invented by **George Cantor** (1845-1918) a famous German mathematician to deal with applied mathematics. Again, it was developed by **Rechard Dedekind** (1831-1916).

The set theory is commonly, employed as ‘foundational system in mathematics’. The language of Set theory can be used in the definitions of nearly all mathematical objects.

Definition

The German word “Menge”, rendered as “Set” in English was coined by **Bernard Balzano** (1781–1848) in his work “the Paradoxes of the infinite degree”. According to **George Cantor** “A set is a gathering together into a whole of definite distinct objects of our perception or of our thought–which are called elements of sets.

In other words, “Set is a group of certain elements of well defined object, group of certain elements of a well defined object, enumerated or descripted according to certain distinctive characteristics through a

notation or a diagram, designated by some capital letter and capable of various operations and algebraic treatment.''

In short, it may be defined as *''a group of objects of certain homogeneous character.''*

Characteristics

From the above definition, the essential characteristics of a set may be analysed as under:

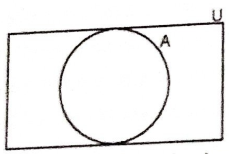
***(i)* It consists of certain elements called members of the set.** The elements may or may not be available or possible in reality. For example, a group of pupils in a school (possible), and a group of immortal persons (impossible). Further, the elements may be enumerated or descripted either in a limited or an unlimited manner. For example, a group of natural numbers up to 10 (limited manner) and the collection of all sand particles in a river (unlimited manner). Furthermore, the elements may be either in an orderly manner, or in a scattered manner each being subscripted by a comma to be distinct from the other.

***(ii)* It refers to a well defined object.** A group of certain elements to constitute a set must relate to an object which can be defined in clear terms of its distinctive characteristics without leaving room for any ambiguity or confusion. For example, a group of Economics Honours students of a College constitutes a set, because it is a case of some elements *i.e.*, students of well defined object, *i.e.*, the Economics honours students of a particular college. But when we speak of ''a group of big rivers'' it does not constitute a set because, it is a group of ambiguous elements, *i.e.*, big rivers, the bigness of which is not defined. However, if the bigness of a river is defined in terms of certain length, then a number of rivers having that magnitude of length will constitute a set of big rivers. Similarly, ''a collection of great men'' will not constitute a set until the object 'greatness' is clearly defined.

***(iii)* It is expressed either through a notation (i.e., a symbolic model), or through a diagram (i.e., Venn diagram). The model used for the purpose, takes the form of a bracket or a Venn diagram exhibited as under :**

(a) A = {x | $x \in$ this or that}, read as A is a set of all x such that x belongs to a particular object.

(b)



Where, U represents a universal set and A, a set of certain elements within the said universe.

***(iv)* It is capable of various operations and algebraic treatment.** A group of enclosed elements to amount to a set proper must be of such nature that it is capable of various treatments like, set operations, set algebra and set applications.

***(v)* It is denoted by a capital letter, viz., A, B etc.** The following is a list of standard sets that are denoted by the appropriate capital letters as follows :

N = Set of natural numbers : {1, 2, 3, ... }

I = Set of integers : {0, ± 1, ± 2, ± 3, ...}

I^+ = Set of positive integers : {1, 2, 3, ...}

I^- = Set of negative integers : {–1, –2, –3, ...}

R = Set of real numbers : {x : x is a real number}

R^{+} = Set of positive real numbers : {x : $x \in$ positive real numbers}

R^- = Set of negative real numbers : {x : $x \in$ negative real numbers}

Q = Set of rational numbers : $\left\{\frac{p}{q} : p \in I \wedge q \in I^+ \vee I^- ...\right\}$

Q^+ = Set of positive rational numbers : $\left\{\frac{p}{q} : p \in I \wedge q \in I^+\right\}$

Q^- = Set of negative rational numbers : $\left\{\frac{p}{q} : p \in I \wedge q \in I^-\right\}$

C = Set of all complex numbers : $\{x : x = x + iy, \wedge x \in R \wedge y \in R\}$

2. METHODS OF PRESENTATION OF A SET

Broadly speaking, there are two methods of presenting a set.

They are :

(*i*) Enumerative method, and (*ii*) Descriptive method.

These are depicted here as under :

(i) Roster Method or Enumerative Method

This method is also called in various other names, *viz.* Roster Method, Tabular Method, Exhaustive Method etc. Under this method, a set is presented by enlisting distinctively all the elements of the concerned object individually within two curly brackets.

The examples of such a presentation are :

(a) A set of vowels or V = {a, e, i, o, u }.

(b) A set of English alphabet or A ={$a, b, c, d....z$}

(c) A set of fundamental digits or D ={0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

(d) A set of first four Presidents of India or

P = {Rajendra Prasad, S. Radhakrishnan, Zakir Hussain, V.V. Giri}

It may be noted that the above method of set presentation is possible, and rather preferable only when, the number of elements to be represented is very few. But where the number of such elements is very large or unlimited, the other method explained as below should be preferred to.

(ii) Set Builder Method or Descriptive Method

This method is also variously called as Descriptive method, Property builder method, Selector method , Rule Method etc. Under this method a set is presented by citing the distinctive characteristic or property of the elements belonging to a defined object within two braces of any form, preferably the curly brackets. The symbol ': or |' is read as 'such that'

The examples of such presentation are :

(i) A set of vowels : V = {x : $x \in$ a vowel in English}

(ii) A set of prime numbers : P = {x : $x \in$ a prime number}

(iii) A set of natural numbers within 100 : N = $\{x : x \in N \wedge 1 \le x \le 99\}$

(iv) A set of professors without a doctoral degree ϕ = {x | $x \in$ Professors $\wedge$ $x \notin$ doctorates}

(v) A set of prime ministers of India belonging to North or South regions

$P = \{x : x \text{ is a prime minister of India} \wedge x \in \text{North} \vee x \in \text{South}\}$

Identification of the symbols used

The various symbols that are used in the above method of set- presentation may be identified brief as under :

(a) V, N, ϕ, P etc. are the English capital letters. They stand for the name or denotation of a set.

(b) { } are the curly brackets which are used as the two enclosures within which a set is described or detailed. Though other forms of enclosures like, parantheses, square brackets or bars may b used but conventionally, only curly brackets are used for the purpose.

(c) x is a lower case letter in English which represents the elements of an object referred to in the set. It may be stated either in an enumerative form as 1, 2, 3 etc. or in a descriptive form as, a natural numbers less than 20. Further, it may be expressed either in an exhaustive manner as (1, 2, 3, or in a continuous manner, as 1, 2, 3, ... n.

(d) $\in$ is a capital epsilon in Greek alphabet which denotes the expression, 'belongs to'.

(e) $\notin$ is a crossed capital epsilon in Greek alphabet which denotes the expression 'does not belong to'

(f) | is a vertical bar and : is a colon. These are used to denote the expression, ''such that''.

(g) $\wedge$ is a symbol of conjunction, which denotes the word, 'And' in a set.

(h) $\vee$ is a symbol of disjunction which denotes the word, 'Or' in a set.

It may be noted that the above method of set presentation is used only when the number of element belonging to a set is either very large or infinite.

3 CARDINALITY OF A SET

By cardinality of a set we mean the number of elements or members contained in a set notwithstanding the repetition of any of them. It is denoted by the denoting letter of a set enclosed by two bars. Thus, the cardinality of the set A will be denoted by |A|, that of B by |B| and that of ϕ by $|\phi|$ etc. If the set A = {1, 2, 3, 4} then its cardinality would |A| = 4, and the cardinality of its power set would be given by $|P(A)| = 2^n = 2^4 = 16$. It may be noted that the cardinality of a null set is zero and of its power se = 1. This is because, $\phi = \{\}$ and $P(\phi) = 2^n = 2^0 = 1$.

4. TYPES OF SET

1. Void set or Empty Set

A set that does not contain nor can invoke any element according to the property indicated in its notation is called a Void, Empty or Null set. It is denoted by the Greek letter ϕ (phi). In Roster method, ϕ is denoted by { } which means A is an empty Set, if the statement $x \in A$ is not true for any x.

The examples of such a set are :

(a) $x \in R \mid x^2 = -2$ is an empty set or ϕ set.

(b) A set of the elements, x which is a perfect square of an integer, $10 < x < 15$ *i.e.*, I = { }

(c) A set of Professors without any academic qualification, *i.e.*, P = { }

2. Valid Set

A set that contains or attracts at least one element according to the property described in its notation is called a valid or full set. It is denoted by a capital letter other than the Greek letter ϕ.

The examples of such a set are :

(a) A set of the present Presidents of the Union of India *i.e.* P = {Ram Nath Kovind}

(b) A set of judges of the Supreme Court of India, *i.e.*, J = {A, B, C, D, E ....}

(c) A set of natural numbers *i.e.* N = {1, 2, 3, ... }

3. Finite Set

A set is called a finite set if it is either a void set or its elements can be listed or counted by natural numbers. **In other words, in a set there are definite number of elements is called a finite set.** It is denoted by a capital letter other than ϕ.

The examples of such sets are :

(*a*) A set of vowels in English : V = {*a, e, i, o, u* }

(*b*) A set of even natural numbers less than 100.

(*c*) A set of commerce students in a college C = $\{x : x \text{ is a commerce student} \wedge x \in \text{a college}\}$

(*d*) A set of positive integers less than 50 : $I^+ = \{x \mid x \in I^+ \wedge x < 50\}$

4. Infinite set

A set whose elements cannot be listed or counted by the natural numbers. Thus, set that attracts innumerable or uncountable number of elements according to the rule depicted in its notation is called an infinite set. It is denoted by a capital letter other than ϕ.

The examples of such a set are :

(a) Set of all points in a plan :

(b) N = {1, 2, 3, ...}

(c) $L = \{x \in R \mid 0 < x < 1\}$

5. Singleton Set

A set consisting of a single elements is called singleton set. Thus, set that consists of only one element or attracts only one element according to its narration is called a singleton set. It is denoted by a capital letter other than ϕ.

The examples of such a set are :

(*a*) The set of the sun : S = {The Sun}

(*b*) The set { 5 } is a singleton set.

(*c*) The set of a natural number $124 < x < 126$: N = {125}

(*d*) The set of a zero : O = {0}, [zero is also an element]

6. Multiton Set

A set that embraces or attracts more than one element according to its narration is called a multiton set. It is represented by a capital letter other than ϕ.

The examples of such a set are :

(a) {0, 1}

(b) $\{a, b, \ldots, z\}$

(c) $\{x \mid x$ is a member of the central board of direct taxes$\}$

7. Equal Set

A set in relation to another set is said to be equal if all the elements (irrespective of t repetitions) belonging to the set, also, belong to that another set. Thus, A = B, if and only if $x \in$ A, $x \in$ B. This is denoted by the sign = placed between the two sets as A = B, or C = D etc.

The examples of such sets are :

(i) A = B, where, A = $\{1, 2, 3\}$ and B = $\{1, 1, 2, 3, 2, 3\}$

(ii) A = B, where, A = $\{r, a, m, a, n, i, m, a, n, i\}$ and B = $\{r, a, m, a, n, i\}$

(iii) P = Q, where, P = $\{2, 3\}$, and Q $\{x : x^2 - 5x + 6 = 0\}$

(iv) A–B, where A= $\{c\ h\ a\ r\ m\}$, B $\{m\ a\ r\ c\ h\}$

8. Equivalent Set

Two finite sets A and B are said to be equivalent it their cardinal numbers are same *i.e.* n (A) = n (B). Thus, set is said to be equivalent or similar to another set, if the number of its eleme including the repeated ones (*i.e.*, its cardinality) is equal to the number of elements of that another *i.e* there is one to one correspondence between the elements of the given sets. Further, any two sets are sai be in one to one correspondence when each member of a set is matched with one and only one membe the other set. The symbol ≡ or ~ is used to denote equivalency of the sets. Thus, if A is equivalent t then it is stated as A ≡ B, or A ~B.

The examples of such sets are :

(i) A = $\{a, e, i, o, u\}$ and B = $\{1, 8, 27, 64, 125\}$

(ii) A = $\{x : x \in$ +3 1st, Economies Hons. class of 64 students$\}$ and
B = $\{x : x \in$ +3 2nd, Economies Hons. class of 64 students$\}$

(iii) F = $\{x : x$ is a letter of the word food$\}$ and
G = $\{x : x$ is a letter of the word Fuel$\}$

It may be noted that while some equal sets are equivalent, some others are not and *vice versa.*

Examples :

(i) If A = $\{1, 2, 3\}$, and B = $\{3, 2, 1\}$, then A = B, so also, A ≡ B

(ii) If A = $\{a, e, i, o, u\}$, and B = $\{a, e, i, o, u, a, e\}$, then A = B but A ≡/ B

(iii) If A = $\{s, a, t\}$ and B = $\{c, a, t\}$, then A ≡ B but A ≠ B.

5. SUBSET

A set is said to be a subset of another set and *vice versa*, if every element of the set is also element of that another set and *vice versa*. If, there are two sets say, A and B and if every element o is an element of B, then A is called a subset of B. If 'A is a subset of B' we write A ⊆ B or 'A is contai in B'.

Thus, A ⊆ B if $a \in$ A ⇒ $a \in$ B

The symbol ⇒ stands for implies.

If A is not a subset of B such relationship will be expressed by A ⊄ B and B ⊄ A.

Example.

(*a*) if A = $\{1, 2, 3\}$ and B = $\{1, 2, 1, 3, 2\}$

(*b*) or A = $\{f, l, o, w\}$, B = $\{w, o, l, f\}$

(*c*) or A = $\{r, a, m, a, n, i\}$ and $\{$B = $r, a, m, a, n\}$

Then A ⊆ B and so also B ⊆ A.

But, (*d*) if A = $\{r, a, m, a, n, i, y, a\}$ and B = $\{r, a, m, a, n, i\}$

Then A ⊄ B and so also B ⊄ A.

Characteristics

There are certain remarkable characteristics of the subsets which may be noted as under:

(a) **All equal sets are subsets of each other,** vide the examples under (*a*), (*b*) and (*c*) given above.

(b) All inequal sets can not be subsets of each other vide the example under (*d*) above.

(c) Some equivalent sets may be subsets of each other but not all. For example,
if A = $\{m, a, r, c, h\}$ and B = $\{c, h, a, r, m\}$ then
A⊆B and B ⊆ A but if A = $\{a, e, i, o, u\}$ and B =$\{1, 2, 3, 4, 5\}$,
Then A ⊄ B and B ⊄ A.

(d) Every set is trivially a subset of itself.
For example, when A = $\{a, c, t\}$ and A = $\{a, c, t\}$ then A ⊆ A

(e) **If A ⊆ ϕ, then A = ϕ**

(f) The number of subsets in a finite set is obtained by **C(*n*) = 2^n** , where n is the no. of elements of a set.

Super set

A set is said to be the super set of another set if that another is a subset of the set under consideration. Thus, a super set is a counterpart of its subset. The relationship of super set is denoted by the symbol ⊇, and its negation by ⊉ .

Thus, if A ⊆ B, then B ⊇ A ⇒ B ⊆ A ⇒ A ⊇ B ⇒ A = B and B=A

It is to be noted that all the equal sets are subsets of each other and, all the equal sets are Super sets of each other.

Proper subset

A set is said to be **proper subset** of another set, if every element of the set under consideration is an element of that another set but not *vice versa*. The relation of proper subset is represented by the symbol ⊂ and its negation by ⊄

Thus,

(i) if A = $\{1, 2, 3\}$ and B = $\{1, 2, 3, 4\}$ then A ⊂ B

(ii) if A = $\{c, a, t\}$ and B = $\{c, a, t, t, l, e\}$ then A ⊂ B
and

(iii) if X = $\{x : x$ is a man$\}$
and Y = $\{x : x$ is a mortal$\}$ then X ⊂ Y

Characteristics

A proper subset has a good deal of useful characteristics some of which may be listed as under :

(a) **Of the two equal sets, there can not arise a proper subset.**

(b) Of the two unequal sets, the smaller one would be the proper subset as all its elements are in larger set but not *vice versa*.

(c) **Of the two equivalent sets there can be a proper subset if the equivalents are not disjoint**

Thus, if $A = \{2, 3, 3\}$ and $B = \{1, 2, 3\}$,

then $A \equiv B$ and also $A \subset B$

But if $A = \{2, 3, 4\}$, and $B = \{4, 5, 6\}$,

then $A \not\subset B$ for A and B are disjoints.

(d) A null set is initially a proper subset of every set except the null set itself.

For example, if $A = \{(1), (2), (3), ( )\}$,

then, ϕ, *i.e.*, ( ) is a proper subset of the set A.

This is represented as $\phi \subset A$.

But a $\phi = \phi$ and so $\phi \subseteq \phi$

but a $\phi \not\subset \phi$

(e) The number of proper subsets in a finite set is obtained by

$$\boxed{n(\subset) = 2^n - 1]}$$

where, n represents the number of elements in a finite set and $n(\subset)$ represents the number of pro subsets of the said set.

Examples :

(a) Let $A = \{1, 2, 3, 4\}$

The various proper subsets that can be formed out of the above set, A are :

(1), (2), (3), (4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4) and ϕ

The total number of subsets thus obtained = 15

This is also corroborated by $n(\subset) = 2^n - 1$

$= 2^4 - 1 = 15$ (where, n = number of elements in the given set, *i.e.*,

Note. It may be noted that the number of proper subsets cannot be determined in case of an infinite since the number of elements in it is infinite.

Proper Superset

A set is said to be a **proper superset** of another set, if that another set is the proper subset of the under consideration. Thus, a superset is a counterpart of its subset. The relation of a superset is given $\supset$ and its negation by $\not\supset$.

Thus, if $A \subset B$, then $B \supset A$

$\Rightarrow$ if $B \subset A$, then $A \supset B$

For example, if $A = \{c, h, a, r, m\}$ and $B = \{h, a, r, m\}$, then B is a proper subset of A a consequently, A is a proper superset of B.

> **Remember** –1. Every set is a subset of itself.
> 2. The empty set is a subset of every set.
> 3. The total number of subsets of a finite set contains n elements is 2^n.

6. FAMILY SET

A family set is one in which the different possible subsets (including the proper subsets) of a given set are shown as the elements of the set. Such a set, is also, otherwise called a set of the sets.

The examples of such sets, represented by the letter, F are :

(i) Given $A = \{1, 2, 3\}$, $F = \{(1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3)\ \phi\}$

(ii) Given $X = \{x, y\}$, $F = \{(x), (y), (x, y)\ \phi\}$

7. POWER SET

Let A be a set, then the collection or family of all subsets of A is **called Power set of A** and is denoted by P(A). Thus, power set is nothing but a family of sets or a set of sets elucidated as above in which the different possible subsets (including the proper subsets) of a given set are shown as its members.

The number of elements of such a set is computed by $\boldsymbol{n}$**.P(S)** $= 2^n$, where, n represents the number of elements in the given set for which the power set is formed.

Thus, if the number of elements in the set, A = 2, the number of elements in its power set would be 2^2 or 4.

EXAMPLE. Let $M = \{x \mid x$ is a month beginning with J$\}$

Representing the elements under the Roster method we get,

A = {January, June, July}

Here, the number of elements in the given set, or $n = 3$

Hence, the number of elements in the power set of the set, A would be given by

$n.P(A) = 2^n = 2^3 = 8.$

This is corroborated as under :

(Jan.), (June), (July), (Jan., June)., (Jan., July), (June, July), (Jan., June, July) $\phi = 8$

As such the required power set would be constituted as under :

P(M) = {(Jan.), (June), (July), (Jan., June), (Jan., July), (June, July), (Jan., June, July), ϕ}

8. UNIVERSAL SET

A universal set refers to the totality of a set constructed upon a particular object. Such a set cannot be a proper subset of any other set but all other sets constructed out of it can be its proper subsets or simply a subset. This set connotes the periphery beyond which any other set cannot be constructed upon the same object. It is denoted by the capital letter, U. It may be a finite or an infinite one.

The examples of such a set may be given as under :

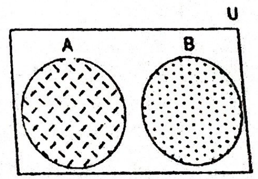
(i) $U = \{x \mid x$ is an integer$\}$

where, $I^+ = \{x : x$ is a positive integer, and

$I^- = \{x : x$ is a negative integer$\}$

(ii) $U = \{x : x \in$ a deck of playing cards$\}$

(iii) $U = \{x : x \in$ English alphabets$\}$



It may be noted that if a universal set is constituted differently, the solution to the same may be different.

For instance, if the universal set is taken to be a set of natural numbers, then $x + 5 = 0$ cannot be correct and the said set will amount to a void one. But if the universal set is chosen to be a set of integers (both +ve and –ve),

then the set of $x + 5 = 0$ may be correct for that x may be equal to –5, and in that case the universal set valid one.

A universal set can be better represented through a Venn diagram as under :

In the said diagram, the rectangle represents the universal set, U and the two circles, A and B repre the two subsets of U under consideration.

9. COMPLEMENTARY SET

A set that represents all the elements of a universal set excluding the elements represented by set thereof is called the complementary set of the said set. If the given set is A, then its complement set is denoted by A', A^c, $\bar{A}$ or ~ A.

Symbolically, it is represented by

(i) $A' = U - A$

or $A = \{x \mid x \in U \text{ but } x \notin A\}$

Thus, if $U = \{0, 1, 2, 3, 4, 5\}$

And $A = \{0, 1, 5\}$

Then the complementary set of A or $A' = U - A = \{2, 3, 4\}$

10. OVERLAPPING SETS

Any two or more sets that have some elements in common are called overlapping sets.

Thus, if $A = \{0, 1, 2, 3\}$, $B = \{2, 3, 4, 5, 6\}$, and $C = \{9, 10\}$, then the two sets A and B which h the elements 2, and 3 in common are called overlapping sets and C will be called a disjoint set.

11. CARTESIAN SET

A set that is constructed with all the ordered pairs whose first elements belong to the first se and second elements to the second set B is called a Cartesian set.

Such a set is denoted by $A \times B$ (read as A cross B) and is given by the model :

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

Example :

(*i*) If $A = \{1, 2, 3, 4\}$, and $B = \{1, 8, 27, 64\}$,

Then $A \times B = \{(1, 1) (1, 8) (1, 27), (1, 64), (2, 1), (2, 8), (2, 27), (2, 64), (3, 1), (3, 8), (3, 27), (3, 64), (4, 1), (4, 8), (4, 27), (4, 64)\}$

However, if the first set is B, and second set is A, then the Cartesian set would be given by the mo

$$B \times A = \{(x, y) . x \in B, y \in A\}$$

Example :

If $B = \{4, 5\}$, and $A = \{a, e, i, o, u\}$,

Then $B \times A = \{(4, a), (4, e), (4, i), (4, o), (4, u), (5, a), (5, e), (5, i), (5, o), (5, u)\}$

EXERCISE (A)

1. Answer the following questions.

(i) What is a set ?

(ii) What is subset ?

(iii) What is complementary set ?

(iv) What are the methods to express a set ?

2. Define the following with illustrations :

(i) Void Set, (ii) Valid Set, (iii) Finite Set, (iv) Infinite Set, (v) Singleton Set, (v) Multiton Set.

3. Explain the following with examples :

(i) Equal Set, (ii) Equivalent Set, (iii) Super Set, (iv) Proper Set.

4. Explain the following along with the characteristics and examples :

(i) Sub-set, (ii) Proper sub-set, (iii) Complimentary Set, (iv) Cartesian Set.

5. Define the following along-with their examples :

(i) Family Set, (ii) Power Set, (iii) Universal Set (iv) Disjoint Sets.

6. Explain the following sets by the set builder method :

(i) Set of all colleges in Orissa

(ii) Set of all letters in English alphabets

(iii) $\{1, 8, 27, 64, 125\}$

(iv) Set of fractions between 7 and 8

(v) Set of odd numbers between 1 and 50

(vi) Set of even numbers between 5 and 10

(vii) Set of natural numbers between 1 and 100

[**Ans.** (*i*) $\{x : x$ is a college in Orissa$\}$ (*ii*) $\{x : x$ is a letter in English alphabets$\}$ (*iii*) $\{x : x \in N\}$, (*iv*) $\{x : x \in$ fraction, $7 < x < 8\}$ (*v*) $\{x : x$ is an odd number, $x \in N, 1 < x < 50\}$ (*vi*) $\{x : x \in N, 5 < x < 10\}$ (*vii*) $\{x : x \in N, 1 < x < 100\}$]

7. Express the following by Roster Method.

(i) $A = \{x : x$ is an even number$\}$

(ii) $B = \{x : x$ is a day in a week$\}$

(iii) $C = \{x : x \in I, x < 10$

(iv) $D = \{x : x$ is a vowel in English alphabet$\}$

8. Which of the following are finite sets and which are infinite sets ?

(i) A set of irrational numbers

(ii) A set of people in Sambalpur district.

(iii) $A = \{x : x$ is a real number between 1 and 10 $\}$.

(iv) $B = \{x : 3x - 5 > 7\}$

[**Ans.** (ii) (iv) Limite (i) and (iv) infinite]

9. Choose the members of the sets from the list attached thereto :

(i) The set of odd numbers between 2 and 10 :

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

(ii) The set of even numbers between 10 and 20 :

11, 12, 13, 14, 15, 16, 17, 18, 19, 20

(iii) The set of Prime numbers between 3 and 21 :

3, 5, 7, 9, 11, 13, 15, 17, 19, 21

[**Ans.** (*i*) 3, 5, 7, 9 ; (*ii*) 12, 14, 16, 18 ; (*iii*) 5, 7, 11, 13, 17, 19]

10. **If A = {2, 3, 4, 5}**

Pick up the true statements of the following :

(*i*) $4 \in A$ (*ii*) $\{3, 5\} \subset A$

(*iii*) $\{2, 1, 5\} \in A$ (*iv*) $\{3, 5, 2, 4\} \neq A$

(*v*) $\{2, 3, 4\} \subseteq A$ (*vi*) $3 \subset A$

(*vii*) $\{2, 4\} \in A$

[Ans. (*i*), (*ii*)]

11. **State which of the following in a null set. Give reasons for your answers :**

(*i*) $A = \{x : x > 1 \text{ and } x < 1\}$; (*ii*) $B = \{x : x + 3 = 3\}$;

(*iii*) $C = \{\phi\}$ (*iv*) Collection of real roots of $x^2 + 1 = 0$

[Ans. (*ii*) and (*iv*)]

12. **Describe the following sets by Roster Method :**

(i) Set of vowels

(ii) Set of integers in $1 < x < 10$

(iii) Set of all positive integers divisible by 5 in $-10 \le x \le 20$

(iv) $\{x : 6x^2 + 11x - 10 = 0, x \in Q\}$, Q being the set of rationals.

(v) $\{x : x \text{ is an integer}\}$

13. **Classify the following statements as true or false :**

(i) $\{0\} \subset \{0, 1, 2\}$

(ii) $\{5\} \subset \{2, 4, 5\}$

(iii) $\phi \subset \{\phi\}$

(iv) A proper subset of a finite set is equivalent to the set

(v) The power set of a given set is the set of all subsets of the set

(vi) Every subset of an infinite set is finite

(vii) Every subset of an infinite set is infinite

(viii) Every subset of a finite set is finite

[Ans. True : (*i*), (*iii*), (*v*), and (*viii*), False : the rest]

14. **How many subsets and proper subsets have a set of (*i*) 3 elements, (*ii*) 5 elements, and (*iii*) *n* elements**

[Ans. (*i*) 8, 7 ; (*ii*) 32, 31 ; (*iii*) 2^n, 2^n-1]

15. **If A = {1, 2, 3}, then construct the power set of A :**

[Ans. P(A) = {ϕ, (1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3)}]

16. **Which of the following sets is a universal set for the other four sets :**

(*i*) Set of integers, (*ii*) Set of natural numbers

(*iii*) Set of negative natural numbers, (*iv*) Set of even natural numbers

(*v*) Set of odd natural numbers

[Ans. (*ii*)]

12. VENN DIAGRAM

This method of presenting a Set was first introduced by Swiss Mathematician **Leonhard Euler.** Later on, the famous English Logician. **John Venn** (1834-1883), brought this idea of diagrammatic presentation

and hence it is named after his name as Venn method or Venn diagram method. Under this method, a square or a rectangular diagram denoted by the letter U is drawn in a free hand manner to represent in its interior part all the elements of an extensive set called Universal set. Then a number of circular diagrams or closed curves, as many as required are drawn in a freehand manner within the said rectangular diagram to represent within their interior parts the elements of all other narrower or intensive sets denoted by the capital letters like, A, B, C, D etc.

The Venn diagrams are invariably used in the operations on sets as depicted below.

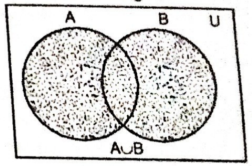
13. SET OPERATION

By set operation we mean mutual application or use of certain existing sets in a manner to produce another new set as desired.

The most popular set operations that are usually performed in the set theory are explained here as under :

(i) Union of Sets

Venn Diagram



The union of two sets A and B, written as **A ∪ B**, is the set of all the elements which belong either to A or to B or to both A and B.

Symbolically, it is represented by

$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in A \text{ and } B\}$

$A \cup B$ is read as 'A union B' or A cup B.

Example : If A = {1, 2, 3, 4, 5} and B = {2, 3, 7, 5, 4, 9}

Then A ∪ B = {1, 2, 3, 4, 5, 7, 9}

The operation of union can be represented by the Venn-diagram as above.

Properties of the Union of sets

(a) It satisfies the **commutative law** in all the cases *i.e.*,

$$A \cup B = B \cup A$$

(b) It satisfies the associative law in all the cases *i.e.*,

$$(A \cup B) \cup C = A \cup (B \cup C)$$

(c) It satisfies the **idempotent law** in all the cases

$$A \cup A = A$$

(d) It satisfies the **distributive law** over intersection of the sets *i.e.*,

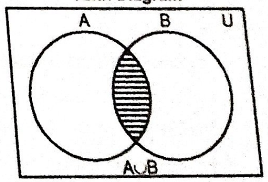
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(e) It satisfies the **identity law** in case of a null set only *i.e.*, $A \cup \phi = A$

(f) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

(ii) Intersection of sets

Venn Diagram



The intersection of two sets A and B, written as **A ∩ B,** is the set of all elements which are common to both A and B.

Symbolically, it is represented by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$A \cap B$ is read as 'A intersection B' or 'A cap B'.

Example : If A = {1, 2, 3} and

B = {2, 3, 4, 5}

then A ∩ B = {2, 3}

The operation of intersection can be represented by the venn diagram as above :

Properties of Intersection

(i) It satisfies the **cumulative law** in all the cases *i.e,*

$A \cap B = B \cap A$

(ii) It satisfies the **associative law** in all the cases *i.e,*

$A \cap (B \cap C) = (A \cap B) \cap C$

(iii) It satisfies the **distributive law** over union of sets *i.e,*

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) It satisfies the **identity law** in case of a null set only *i.e.,*

$A \cap \phi = \phi$

(v) It satisfies the **idempotent law** in all the cases *i.e.,*

$A \cap A = A$

(vi) $A \cap B \subset A$ and $A \cap B \subset B$

(iii) Disjoint Sets

If two sets A and B have no elements in common, *i.e.,* if no element of A is in B and no element of B is in A, then A and B are said to be mutually exclusive or disjoint sets.

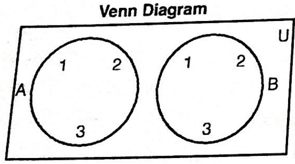
Symbolically, it is represented by

$\mathbf{A \subset B = \phi}$, when A and B are disjoint.

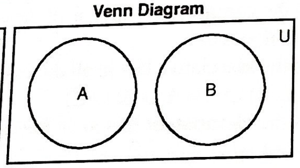
Example. If A = {1, 2, 3} and B = {4, 5, 6}, then they are disjoint sets as they have no common elements.

The above operation can be represented by the Venn diagrams as under :

Venn Diagram



Venn Diagram

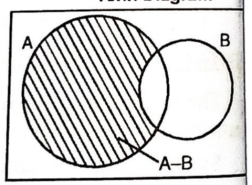


The two disjoint sets A and B having no common elements in them are shown in the Venn diagram above.

(iv) Difference of Two Sets

The difference of two set A and B is the set of elements which belong to A but which do not belong to B. We denote the difference of A and B by **A–B or A / B.**

Venn Diagram



Symbolically, it is represented by

$A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$

Example. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8, 9\}$, then

$A - B = \{1, 2, 3\}$ and

$B - A = \{6, 7, 8, 9\}$.

The above operation of difference of two sets can be represented by the Venn diagram as thus :

Properties of difference-operation

(a) If A, B and C be any three sets, then $A - (B \cup C) = (A - B) \cap (A - C)$

(b) If A, B and C be any three sets, then $A - (B \cap C) = (A - B) \cup (A - C)$

(c) If A and B be any two sets, then $A - (A - B) = A \cap B$

(d) If A and B be any two sets, and $A \cap B = \phi$, then $A \backslash B = A$.

(e) If A and B be any two sets, then $A \cap (B \backslash A) = \phi$

(f) If A and B be any two sets, then $(A \backslash B) = A - B$

(g) If A, B be any two sets, then $A' - B' = B - A$

If A, B and C be any three sets, then $A \cap (B - C) = (A \cap B) - (A \cap C)$

(a) If A, B and C be any three sets, then $A \cap (B - C) = (A \cap B) - C$

(b) If A, B and C be any three sets, then $A \cup (B - C) \neq (A \cup B) - A \cup C$

(c) If A and B be any two sets, then $(A - B) \cap B = \phi$

(d) If A and B be any two sets, then $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

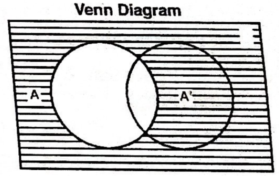
(v) Complement or Negation of a set

This is always expressed with respect to the Universal set U. **The complement of a set A is the set of all the elements of the Universal set U which is not belong to A *i.e.,* it is the difference of the universal set U from the set A.** We denote the complement of the set A by A^c or A' or ~ A.

Symbolically, it is represented by

$A' = \{x : x \in U \text{ but } x \notin A\}$

Venn Diagram



Example. If

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{3, 4, 5\}$ then

$A' = U - A = \{1, 2, 6, 7, 8, 9, 10\}$

The operation of complement of set can be represented by a venn diagram as at the right :

The complementation operation has two important properties as per **De-Morgan's Law.**

These are depicted here as under :

(a) $\mathbf{(A \cup B)' = A' \cap B'}$

(b) $\mathbf{(A \cap B)' = A' \cup B'}$

Note : *(i)* $A \cap A' = \phi$ *(ii)* $A \cup A' = U$

(iii) $U' = \phi$ *(iv)* $\phi' = U$

(v) $(A')' = A$

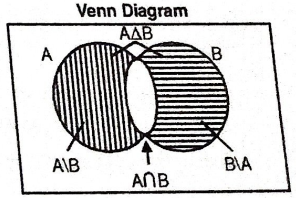
(vi) Symmetric Difference

The symmetric difference of two sets A and B, written as A Δ B, **is the set of all elements which belong either to A–B, or to B–A , but not to $A \cap B$, *i.e.,* $(A - B) \cup (B - A)$ or $(A \cup B) - (A \cap B)$.**

Symbolically, this can be represented by

$A \,\Delta\, B = \{x : x \in (A - B) \text{ and } x \in (B - A)\}$

Venn Diagram



Example. If $A = \{1, 2, 3, 4\}$

And $B = \{2, 3, 5, 6\}$

Then $A \Delta B = (A - B) \cup (B - A)$

$= \{1, 4\} \cup \{5, 6\} = \{1, 4, 5, 6\}$

The above operation of symmetric difference of the sets can be represented by the Venn diagram at the right.

Properties

(i) It is **commutative** in nature *i.e.*, $A \Delta B = B \Delta A$

(ii) It is **associative** in nature *i.e.*, $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

(iii) It cannot arise in case of two equal sets *i.e.*, $A \Delta A = \phi$

(iv) It satisfies the relation :

(1) $A \Delta (A \cap B) = (A - B)$

(2) $(A \Delta B) \cup (A \cap B) = A \cup B$

ILLUSTRATION 1. If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6, 7\}$,

then find, (*i*) $A \cap B$, (*ii*) $B \cup C$, (*iii*) $A \cap (B \cup C)$, (*iv*) $A \cup (B \cup C)$

SOLUTION

(*i*) $A \cap B = \{1, 2, 3, 4\} \cap \{2, 4, 6, 8\} = \{2, 4\}$

(*ii*) $B \cup C = \{12, 4, 6, 8\} \cup \{3, 4, 5, 6, 7\} = \{2, 3, 4, 5, 6, 7, 8\}$

(*iii*) $A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{2, 3, 4, 5, 6, 7, 8\} = \{2, 3, 4\}$

(*iv*) $B \cap C = \{2, 4, 6, 8\} \cap \{3, 4, 5, 6, 7\} = \{4, 6\}$

$\therefore A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{4, 6\} = \{1, 2, 3, 4, 6\}$

ILLUSTRATION 2. If $A = \{3, 4, 2\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 4, 6, 8\}$

Verify that, $A \cap (B - C) = (A \cap B) - (A \cap C)$

SOLUTION

We have, $B - C = \{3, 4, 5, 6\} - \{2, 4, 6, 8\} = \{3, 5\}$

$\therefore$ $A \cap (B - C) = 3, 4, 2\} \cap \{3, 5\} = \{3\}$

Again, $A \cap B = \{3, 4, 2\} \cap \{3, 4, 5, 6\} = \{3, 4\}$

And $A \cap B = \{3, 4, 2\} \cap \{2, 4, 6, 8\} = \{2, 4\}$

$\therefore$ $(A \cap B) - (A \cap C) = \{3, 4\} - \{2, 4\} = \{3\}$

Hence, from (1) and (2) it is verified that

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

ILLUSTRATION 3. If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$ and $C = \{1, 4, 5\}$,

then verify that, $A - (B \cup C) = (A - B) \cap (A - C)$

SOLUTION

We have,

$B \cup C = \{3, 4, 5\} \cup \{1, 4, 5\} = \{1, 3, 4, 5\}$

$\therefore$ $A - (B \cup C) = \{1, 2, 3, 4\} - \{1, 3, 4, 5\} = \{2\}$

Again, $A - B = \{1, 2, 3, 4\} - \{3, 4, 5\} = \{1, 2\}$

And, $A - C = \{1, 2, 3, 4\} - \{1, 4, 5\} = \{2, 3\}$

$\therefore$ $(A - B) \cup (A - C) = \{1, 2\} \cap \{2, 3\} = \{2\}$

Hence, from (1) and (2) it is verified that

$$A - (B \cup C) = (A - B) \cap (A - C)$$

ILLUSTRATION 4. If $A = \{1, 3, 5\}$, $B = \{2, 4, 6, 8\}$, $C = \{2, 5, 10\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

then verify that, $(A \cap B)' = A' \cup B'$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION

(*a*) We have,

$A \cap B = \{1, 3, 5\} \cap \{2, 4, 6, 8\}\} = \phi$

$\therefore$ $(A \cap B)' = U - (A \cap B) = U - \phi = U$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ ...(1)

Again, $A' = U - A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{1, 3, 5\} = \{2, 4, 6, 7, 8, 9, 10\}$

And, $B' = U - B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} - \{2, 4, 6, 8\} = \{1, 3, 5, 7, 9, 10\}$

$\therefore$ $A' \cup B' = \{2, 4, 6, 7, 8, 9, 10\} \cup \{1, 3, 5, 7, 9, 10\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ ...(2)

Hence, from (1) and (2) it is verified that $(A \cap B)' = A' \cup B'$

(*b*) We have

$B \cup C = \{2, 4, 6, 8\} \cup \{2, 5, 10\} = \{2, 4, 5, 6, 8, 10\}$

$\therefore$ $A \cap (B \cup C) = \{1, 3, 5\} \cap \{2, 4, 5, 6, 8, 10\} = \{5\}$ ...(1)

Again, $A \cap B = \{1, 3, 5\} \cap \{2, 4, 6, 8\} = \phi$

And $A \cap C = \{1, 3, 5\} \cap \{2, 5, 10\} = \{5\}$

$\therefore$ $(A \cap B) \cup (A \cap C) = \phi \cup \{5\} = \{5\}$ ...(2)

Hence, from (1) and (2) it is verified that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

ILLUSTRATION 5. If the universal set $U = \{x : x \in N, 1 \le x \le 12\}$, where N is the set of Natural number, and $A = \{1, 9, 10\}$, $B = \{3, 4, 6, 11, 12\}$ and $C = \{2, 5, 6\}$ are subset of U.

then find the set : (*i*) $(A \cup B) \cap (A \cup C)$ (*ii*) $A \cup (B \cap C)$ (*iii*) $(A \cup B \cup C)'$.

SOLUTION

We have, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Thus, (*i*) $A \cup B = \{1, 9, 10\} \cup \{3, 4, 6, 11, 12\}$

$= \{1, 3, 4, 6, 9, 10, 11, 12\}$

And $A \cup C = \{1, 9, 10\} \cup \{2, 5, 6\} = \{1, 2, 5, 6, 9, 10\}$

$\therefore (A \cup B) \cap (A \cup C) = \{1, 3, 4, 6, 9, 10, 11, 12\} \cap \{1, 2, 5, 6, 9, 10\} - \{1, 6, 9, 10\}$

(*ii*) $B \cap C = \{3, 4, 6, 11, 12\} \cap \{2, 5, 6\} = \{6\}$

$\therefore$ $A \cup (B \cap C) = \{1, 9, 10\} \cup \{6\} = \{1, 6, 9, 10\}$

EXERCISE (B)

1. Discuss the following set operations along with their properties : (*i*) Union of Sets, (*ii*) Intersection of Sets, (*iii*) Complementation of a Set, (*iv*) Difference of Sets, (*v*) Multiplication of Sets.
2. Find the following :

(a) (i) $A \cup A'$, (ii) $A \cap \phi$, (iii) $A \cap A'$; (iv) $A \cup B$ when $B \subseteq A$, and

(b) (iv) $A \cap B$, when $A \subseteq B$.

[**Ans.** (i) U, (ii) ϕ, (iii) ϕ, (iv) A, (v) A]

3. Sketch a Venn diagram to represent the three sets, A = {1, 2, 3, 4}, B = {2, 3, 4, 5 C = {3, 4, 5, 6}, and thence locate $A \cap B \cap C$, and $A \cup B \cup C$.

[**Ans.** (i) {3, 4}, (ii) {1, 2, 3, 4, 5, 6

4. Draw a Venn diagram to represent the sets U = {9, 8, 7, 6, 5, 4, 3, 2, 1, 0}, A = {1, 2, 3, 4} a B = {2, 3}, and then, locate $A \cap B$, and $A \cup B$.

[**Ans.** (i) B, (ii) A]

5. If A = {5, 6, 7, 8, 9}, B = {2, 4, 6, 8, 10, 12} and C = {3, 6, 9, 12}, then verify th $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. If U = {*a, b, c, d, e, f, x, y, z, w*}, A = {*a, b, c, d, e*}, B = {*b, d, x, y, z*} and A\B is defined $A \cap B'$, verify that $(A\backslash B)' = A' \cup B$.

7. If A = {3, 2, 1}, B = {5, 4, 3, 2}, and C = {8, 6, 4, 2}, then verify that (i) $A \cup B = (A\backslash B) \cup$ (ii) $A\backslash(A\backslash B) = A \cap B$ and (iii) $A \cap (B\backslash C) = (A \cap B) - (B \cap C)$

8. If U = {1, 2, 3,.... 8, 9}, A = {4, 3, 2, 1} and B = {8, 6, 4, 2}, then construct the following sets: (i) $A \cup B$, (ii) $A \cap B$, (iii) A', (iv) $(A \cup B)'$ and (v) $(A \cap B)'$

[**Ans.** (i) {1, 2, 3, 4, 6, 8}, (ii) {2, 4}, (iii) {5, 6, 7, 8, 9
(iv) {5, 7, 9}, (v) {1, 3, 5, 6, 7, 8, 9

9. If A ={2, 1, 0}, B = {4, 3, 2} and C={5, 4, 2}, then show that

(i) $A-(B \cup C) = (A-B) \cap (A-C)$

(ii) $A-(B \cap C) = (A-B) \cup (A-C)$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(v) $A-(B-C) \neq (A-B)-C$

(vi) $A \cap (B \cap C) = (A \cap B) \cap C$

(vii) $A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$

10. If Δ denotes the symmetric difference of two sets P and Q, then find $P \Delta Q$ for the following :

(i) P = {*a, b*}, Q = {*a, c*}

(ii) P = {*a, b*}, Q = {*b, c*} [**Ans.** (i) {*b, c*}, (ii) {*a, c*

11. If X = {1, 3, 5, 7, 9} and Y = {3, 5, 8}, then find the symmetric difference of the sets X and Y.

[**Ans.** {1, 7, 9}]

12. If U = {4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}

A = {4, 5, 6, 7}

B = {5, 7, 9, 10}

and C = {7, 8, 9, 10, 12, 13}

Explain the following sets :

(i) A', (ii) $(B')''$, (iii) $A' \cup C'$, (iv) A–B, (v) B–A, (vi) $A \Delta C$, (vii) $A \cap B^1$

(**Ans.** (i) {8, 9, 10, 11, 12, 13, 14}, (ii) {5, 7, 9, 10} (iii) {4, 5, 6, 7, 8, 9, 10, 11,12, 13, 14}
(iv) {4, 6}, (v) {9, 10} (vi) {4, 5, 6, 7, 8, 9, 10, 11, 12, 13} (vii) {4, 6, 8, 9, 10, 11, 12, 13, 14

14. SET ALGEBRA

By set algebra we mean the processing of the set operations with its relevant symbols and formulae to arrive at the desired result relating to a problem representable by some sets. The various symbols and signs that are used in the set operations include $\cup, \cap, |, \backslash, \times$ etc. The various formulae or models that are used in the processing of set operations are derived under a set of principles or laws applicable differently to different situations.

A list of such formulae is given here as under :

List of formulae or Laws of Set Theory

1. **Under the Commutative Law**
   (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$
   (iii) $A-B \neq B-A$
2. **Under the Associative Law**
   (i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$
3. **Under the Distributive law**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **Under the Idempotent Law**
   (i) $A \cup A = A$ (ii) $A \cap A = A$
5. **Under the Identity Law**
   (i) $A \cup \phi = A$ (ii) $A \cap \phi = \phi$
   (iii) $A \cup U = U$ (iv) $A \cap U = A$
6. **Under the Complement Law**
   (i) $A' = U-A$ (ii) $(A')' = A$
   (iii) $(A') \cap A = \phi$ (iv) $(A') \cup A = U$
7. **Under De-Morgan's Law**
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Besides the above two models of De-Morgan, there are twelve other models given by De-Morgan relating to difference in operation of sets. These have been cited under the head, "Properties of Difference operation" discussed earlier.

15. SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

If A, B and C are finite Sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$ when A and B are disjoint non-void sets.

(iii) $n(A-B) = n(A) - n(A \cap B)$ *i.e.* $n(A-B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B.

$= n(A-B) \cup (B-A)$

$= n(A-B) + n(B-A)$ [$\because$ (A–B) and (B–A) are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

Thus, $\boxed{n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)}$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) Number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n[(A \cap B)' = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n[(A \cap B)')] = n(U) - n(A \cup B)$

EXAMPLE 1. If A and B be two sets containing 6 and 12 elements respectively, what can be th minimum number of elements in $A \cup B$? Also find the maximum number of elements in $A \cup B$.

SOLUTION. We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. This shows that $n(A \cup B)$ is minimum o maximum according to the elements as $n(A \cap B)$ is maximum or minimum respectively.

When $n(A \cap B) = 0$, $n(A \cap B)$ is minimum

(i) This is a case when Set A and B are disjoint and $A \cap B = \phi$, Here $n(A \cup B)$ is maximum, Th

$n(A \cup B) = n(A) + n(B) - 0$

$= 6 + 12 = 18$

So, the maximum number of elements in $A \cup B$ is 18.

(ii) $n(A \cap B)$ is maximum when $A \subseteq B$ and $n(A \cap B) = 6$, [here $n(A \cup B)$ is minimum]

$\therefore n(A \cap B) = n(A) + n(B) - n(A \cap B)$

$= 6 + 12 - 6 = 12$

So, the minimum number of elements in $A \cup B$ is 12.

EXAMPLE 2. If A, B and C are three sets and U is the Universal set such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$.

Find $n(A' \cap B')$

SOLUTION. We have, $A' \cap B' = (A \cup B)'$

Thus, $n(A' \cap B') = n[(A \cup B)'] = n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 700 - [200 + 300 - 100]$

$= 700 - 400 = 300$

EXAMPLE 3. Two finite sets have 'm' and 'n' elements. The total number of subsets of the first set 56 more than the total number of subsets of the second set. Find the values of 'm' and 'n'.

SOLUTION. Let A and B be two sets having 'm' and 'n' elements respectively.

Thus, Number of subsets of $A = 2^m$

Number of subsets of $B = 2^n$

According to proposition $2^m - 2^n = 56$

or $\quad 2^n(2^{m-n} - 1) = 56$

or $\quad 2^n(2^{m-n} - 1) = 8 \times 7 = 2^3(2^3 - 1)$

$n = 3$ and $m - n = 3$

$n = 3$ and $m = 3 + 3 = 6$

ILLUSTRATION 6. In a town of 10,000 families it was found that 40% families buy news paper A % families buy news paper B and 10% families buy newpaper C, 5% families buy A and B, 3% buy I C, and 4% buy A and C. If 2% families buy all the three news paper, then (i) find the number o ilies which buy A only (ii) the number of families which buy none off A, B and C newspaper.

SOLUTION. We have, $N = 10{,}000$, $n(A) = 40\%$ of $10{,}000 = 4000$

$n(B) = 20\%$ of $10000 = 2000$, $n(C) = 10\%$ of $10000 = 1000$,

$n(A \cap B) = 5\%$ of $10000 = 500$, $n(B \cap C) = 300$, $n(A \cap C) = 400$ and $n(A \cap B \cap C) = 200$.

(i) Here, we have to find out the number of families which buy A newspaper only is,

$n(A \cap B' \cap C' = n[(A) \cap (B \cup C)']$

$= n(A) - n(A \cap (B \cup C)')$

$= n(A) - n[(A \cap B) \cup (A \cap C]$

$= n(A) - n(A \cap B) + n(A \cap B)\, B \cap = n(A \cap B \cap C)$

$= 4{,}000 - \{500 + 400 - 200\}$

$= 4{,}000 - 700 = 3{,}300$

Thus, 3,300 families buy only newspaper A.

(ii) Now, we have to find out the number of families which buy none of the A, B and C newspaper. We have, $n(A' \cap B' \cap C')$

$= n(A \cup B \cup C)'$

$= N - n(A \cup B \cup C)$

$N - [n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)]$

$= 10{,}000 - [4{,}000 + 2{,}000 + 1{,}000 - 500 - 300 - 400 + 200]$

$= 10{,}000 - 6{,}000 = 4{,}000$

Thus, 4,000 families buy none of the A, B and C newspaper.

ILLUSTRATION 7. If A, B and C be any three sets, then

Prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

SOLUTION

Let x be any element of $A - (B \cup C)$

Thus, $x \in A - (B \cup C) \Rightarrow x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in (A - B)$ and $x \in (A - C)$

$\Rightarrow x \in (A - B) \cap (A - C)$

$\therefore \quad A - (B \cup C) \subseteq (A - B) \cap (A - C)$ ...(1)

Again, let y be any element of $(A - B) \cap (A - C)$

Thus $\quad y \in (A - B) \cap (A - C)$

$\Rightarrow y \in (A - B)$ and $y \in (A - C)$

$\Rightarrow (y \in A$ and $y \notin B)$ and $(y \in A$ and $y \notin C)$

$\Rightarrow y \in A$ and $(y \notin B$ or $y \notin C)$

$\Rightarrow y \in A$ and $y \notin (B \cup C)$

$\Rightarrow y \in A - (B \cup C)$

$\therefore \quad (A - B) \cap (A - C) \subseteq A - (B \cup C)$ ...(2)

Hence, $\quad A - (B \cup C) = (A - B) \cap (A - C)$. Proved.

ILLUSTRATION 8. Prove that $(A - B) = A \cap B'$ and hence, show that :

(i) $B \cap (A - B) = \phi$ $\quad$ (ii) $B \cup (A - B) = A \cup B$

SOLUTION

Let x be any element of $A - B$

Thus $x \in (A-B) \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in (A \cap B')$

$\therefore \quad (A-B) \subseteq A \cap B'$

Again, let y be any element of $(A \cap B')$

Thus $y \in (A \cap B') \Rightarrow y \in A$ and $y \in B'$

$\Rightarrow y \in A$ and $y \notin B$

$\Rightarrow y \in (A-B)$

$\therefore \quad (A \cap B') \subseteq (A-B)$

Hence, $A-B = A \cap B'$

(i) L.H.S. $= B \cap (A-B) = B \cap (A \cap B')$

$= A \cap (B \cap B') = A \cap \phi = \phi$ Proved

(ii) L.H.S. $= B \cup (A-B) = B \cup (A \cap B')$

$= (B \cup A) \cap (B \cup B') = (B \cup A) \cap U$

$= B \cup A = A \cup B$. Proved.

EXERCISE (C)

1. (i) Prove that $A \cup B = B \cup A$
   [**Hints.** Show that $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$]
   (ii) Prove that $A \cap B = B \cap A$
   [**Hints.** Show that $A \cap B \subseteq B \cap A$ and *vice versa*]
2. (i) Prove that $(A \cup B) \cup C = A \cup (B \cup C)$
   Also, verify the proof by a Venn diagram.
   [**Hints.** Show that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and *vice versa*]
   (ii) Prove that $(A \cap B) \cap C = A \cap (B \cap C)$
   Also, verify the result through a Venn diagram.
   [**Hints.** Show that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ and *vice versa*]
3. (i) If A, B and C be any three sets, then show that
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
   [**Hints.** Prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and *vice versa*]
   (ii) If A, B and C be any three sets, then prove that
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   Also, test your proof through a Venn diagram.
   [**Hints.** Prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and *vice versa*]
4. (i) If A and B be any two sets, then prove that $(A \cup B)' = A' \cap B'$
   [**Hints.** Show that $(A \cup B)' \subseteq A' \cap B'$]
   (ii) If A and B be any two sets, then show that
   $(A \cap B)' = A' \cup B'$
   [**Hints.** Prove that $(A \cap B)' \subseteq A' \cup B'$ and *vice versa*]
5. (i) If A, B and C be any three sets, then prove that
   $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
   [**Hints.** Show that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$ and *vice versa*]

(ii) If A, B and C be any three sets then prove that
$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
[**Hints.** Show that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and *vice versa*]
(iii) If A, and B be any two sets, then prove that
$A \cup B = (A \setminus B) \cup B$
[**Hints.** Show that $A \cup B \subseteq (A \setminus B) \cup B$ and *vice versa*]
(iv) If A and B be any two sets then show that $A \cap (B \setminus A) = \phi$
[**Hints.** Prove that $A \cap (B \setminus A) \subseteq \phi$]

6. If A and B constitute a universal set where, A = {4, 2, 1, 3}, and B = {7, 5 6, 4, 3, 2}, then find,
   (i) $n(A \cup B)'$ and its form
   (ii) $n(A' \cup B)$ and its form
   (iii) $n(A \cup B')$ and its form
   (iv) $n(A' \cup B')$ and its form
   [**Ans.** (i) 0, ϕ, (ii) 6,{2, 3, 4, 5, 6,7} (iii) 4, { 1, 2, 3, 4}, (iv) 4, {1, 5, 6, 7}]

16. SET APPLICATIONS

ILLUSTRATION 9. In a Economics Honours class of 96 students, only 50 play cricket, and 32 play cricket but not football. Determine through set algebra (i) The number of students who play both cricket and football, and (ii) the number of students who play football but not cricket.

SOLUTION

Determination of the number of students as required by the set algebraic method

Let U represent the total of students, in a Economies Honours class

A, the student playing the cricket

And B, the student playing the football

According to the information we have,

$$n(U) = 98, n(A) = 50 \text{ and } n(A \cap B') = 32$$

Thus, (i) The number of students playing both cricket and football is given by

$$n(A \cap B) = n(A) - n(A \cap B') = 50 - 32 = 18$$

And (ii) The number of students playing football but not cricket is given by

$$(B \cap A') = n(B) - n(A \cap B)$$

Where, $n(B) = n(U) - n(A) + n(A \cap B) = 96 - 50 + 18 = 64$

Thus, $n(B \cap A') = 64 - 18 = 46$.

Hence, the number of students playing both cricket and football = 18 and those playing football but not cricket = 46.

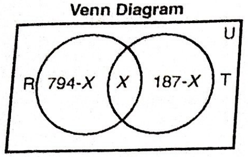
ILLUSTRATION 10. Use Venn diagram is to solve the following problems : "In a statistical investigation of 1003 families of Cuttack, it was found that 63 families had neither a radio nor a T.V., 794 families had a radio and 187 a T.V." How many families in that group had both a radio and a T.V. ?

SOLUTION

Let R be set of families having a radio and T the set of families having a T.V.

Then $n(R \cup T)$ = Number of families having at least one of Radio and T.V. = 1003 − 63 = 940

Venn Diagram



$n(R) = 794$ and $n(T) = 187$

Let $n(R \cup T) = x$, *i.e.*, x families had both a radio and a T.V. Then the number of families who ha only radio $= 794 - x$ and the number of families who have only T.V. $= 187 - x$.

From the above venn diagram we have, $794 - x + x + 187 - x = 940$

or $981 - x = 940$ or $x = 41$

Hence, the required number of families having both a radio and a T.V. = 41.

ILLUSTRATION 11. *Of a group of 200 persons, 100 are interested in music, 70 in photography a 40 in swimming. Furthermore, 40 are interested in both music and photography, 30 in both music a swimming, 20 in photography and music, and 10 in all the threes. How many are interested in photograph but not in music and swimming ?*

SOLUTION

Let M be the set of persons who are interested in music. Similarly, let sets P and S be the perso interested in photography and swimming respectively. Thus,

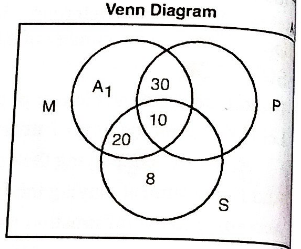
$n(M) = 100, n(P) = 70, n(S) = 40$

Thus, $n(M \cap P) = 40, n(M \cap S) = 30. n(P \cap S) = 20$

and $n(M \cap P \cap S) = 10$

We have to find out, $n(P \cap M' \cap S')$ has been presented through Venn diagramme.

Venn Diagram



Now, $n(M \cap P \cap S') = 40 - 10 = 30$

and $n(P \cap S \cap M') = 20 - 10 = 10$

From the above venn diagram also, we get

$n(P \cap M' \cap S') = n(P)(30 + 10 + 10) = 70–50 = 20$

Hence, the required number of persons who are interested in photography but not in music a swimming is 20.

17. RELATION AND FUNCTION

17. (a) Meaning of Relation :

In ordinary sense, the term 'relation' means some ties between some persons or objects on certa basis like, natural, constitutional, contractual etc.

The examples of such relations are :

(i) A is father of B (natural)

(ii) Bhubaneswar is the capital of Odisha (Constitutional)

(iii) X is the partner of Y (Contractual)

(iv) H is the husband of W (Marital)

But in Mathematics, the term, 'relation' means some logical ties between some numbers or objec

obtained from a set of ordered pairs that consists of two elements, x and y written within parantheses like, (x, y), such that one of them say, x is designated as the first element and the other say, y as the second element.

The examples of such relations are :

(*i*) The natural numbers and their squares

viz. : (1, 1) (2, 4) (3, 9), (4, 16), (5, 25) etc.

(*ii*) The natural numbers and their cube roots

viz. : (1, 1) (8, 2), (27, 3), (64, 4), (125, 5) etc.

(*iii*) The natural numbers and their greater ones

viz. : (3, 5) (4, 7) (5, 11), (7, 9) etc.

In other words, the mathematical relations are the subsets of the sets of Cartesian products of two variables.

Thus, if any two sets are :

(*i*) $A = \{1, 2, 3, 4\}$ and (*ii*) $B = \{1, 4, 9, 16\}$

The Certain product of these two sets would be,

$A \times B = \{ (1, 1), (1, 4), (1, 9), (1, 16), (2, 1), (2, 4), (2, 9), (2, 16), (3, 1), (3, 4), (3, 9), (3, 16), (4, 1), (4, 4), (4, 9), (4, 16)]$

If a set is constructed with the natural numbers and their squares from the above Cartesian product, then the relation from the set A to the set B will be obtained as

$R : A \rightarrow B = \{ (1, 1), (2, 4), (3, 9), (4, 16)\}$

From the above analysis, it must be noticed that the relation (R) is a subset of a Cartesian product *i.e.*, $R \subset A \times B$

It may be noted that the mathematical relations are invariably expressed inc ertain forms like, "it parallel to", "is less than", "is equal to" etc. When any such relations are between any two numbers or objects, they are called Binary relations and are expressed as xRy, aRb etc. But when any such relations are among any three numbers or objects, they are expressed as $xRyRz$, $aRbRc$ etc. The higher order relations of the above nature are accordingly expressed with multiplication signs as indicated above.

Definition

From the above, discussion the term 'relation' can be defined as "a subset of some ordered pairs of certain elements of a cartesian product A×B, where, the first element belongs to the set A and the second element to the set B. and the two elements are related to each other on certain logical basis."

Thus a relation from A to B is symbolically denoted by $R : A \rightarrow B$ or xRy $\{x, y) \mid x \in A, y \in B$ and $x, y \in R\}$ and is read as x related to y = { set of x and y such that x belongs to the set A, y to the set B and both x and y relate to the set R'}

Characteristics

From the above definition, the essential characteristics of a relation may be enumerated as under :

(i) It is a subset of a Cartesian product, *i.e.*, $R \subset A \times B$.

(ii) It consists of some ordered pairs of any two elements which are tied with each other on certain logical basis.

(iii) The first element of each pair belong to the first set, A and the second element belongs to the second set B.

(iv) The collection of all its first elements is called **Domain** of the relation denoted by $d(R)$.

(v) The collection of all its second elements is called **Range** of the relation denoted by $r(R)$.

DIFFERENT TYPES OF RELATION

According to the properties, the relations between any two numbers or objects can be classified in the following nine types :

(i) Reflexive relation

A relation between any two objects is said to be reflexive when it is related to itself. The examples of such relations are :

(a) X gets the same salary as Y, (when both X and Y serve in the same college).

(b) A has the same experience as B (when both A and B belong to the same field of working).

Symbolically, such relations are expressed as xRx, aRa, etc. for all x or $a \in$ A.

(ii) Symmetric relation

A relation between any two objects is said to be symmetric when the expression of the relation remains the same, even if the equation, between the objects is altered.

The examples of such relations are :

(a) X is the brother of Y (same as Y is the brother of X)

(b) A is the partner of B (same as B is the partner of A)

(c) P is 5 metres away from Q (same as Q is 5 metres away from P).

Symbolically, such relations are expressed as $xRy \to yRx$, $aRb \to bRa$, $pRq \to qRp$ etc. for all x and y, a and b or p and $q \to$ A.

(iii) Anti-symmetric relation

When the relation between any two objects is of such nature that when the position of the objects is altered, the relation does not remain the same, it is called anti-symmetric relation.

The examples of such relation are :

(a) X is the son of Y

(b) A is the husband of B

(c) $P < q$.

Symbolically such relations are expressed as $xRy \neq yRx$ for all x and y .A.

(iv) Transitive relation

A relation between any two objects is said to be transitive, if such relation could be extended to another one or more objects.

The examples of such relation are :

(a) X is the contemporary of Y, Y is the contemporary of Z, and so X is the contemporary of Z.

(b) A is parent of B, B is the parent of C and so A is the parent of C.

Symbolically, such relations are expressed as xRy, and yRz .xRz, for all x, y, z .A.

(v) Non-transitive relation

A relation between any two objects is said to be non-transitive, when the said relation being extended to any other object does not remain the same.

The examples of such relation are :

(a) X is the father of Y, Y is the father of Z. For this X can not be the father of Z.

(b) A is the teacher of B, B is the teacher of C. For this, A can not be the teacher of C.

Symbolically, such relations are denoted by xRy, and $yRz \neq xRz$, for all x, y, z .A.

(vi) Equivalence relation

A relation between any two objects is said to be equivalence if it is reflexive, symmetric and transitive as well.

The examples of such as relation are :

(a) X gets the same salary as Y and Z (where all X, Y and Z serve in the same establishment).

(b) A is parallel to B and C, *i.e.* A ∥ B ∥ C Symbolically, such relations are denoted by $xRy \to yRz \to xRz$, for all x, y and $z \in$ Z.

(vii) Order relation

A relation between any two objects is said to be an order one when it is non-reflexive, non-symmetric but only transitive.

The examples of such a relation are :

(a) $X < Y < Z \to X < Z$ (b) $X > Y > Z \to X > Z$

But the relations like $Z \leq Y$ or $X \geq Y$ will not amount to order relations in as much as although they are transitive, they are reflexive as well.

(viii) Binary relation

A relation is said to be a Binary one when it ties only two objects on certain logical basis.

Thus, the relation between 5 and 7, or 3 and 9 are the examples of binary relation. Symbolically, such relation are expressed as xRy, for all x, y . A.

(ix) Inverse relation

An inverse relation is one which is obtained by reversing the positions of any two related objects. Thus, if R, be a relation from A to B, then its inverse relation will be the relation from B to A denoted by R–1. For example, if we have a relation R of the type, > (more than for the sets, A= {3, 5,7}, and B = {2, 4, 6} as R : A → B = {(3, 2), (5, 2), (7, 2), (7, 4) (7, 6)}, the inverse relation R–1 of the type, < (less than) would be as follows :

$R^{-1} : B \to A = \{(2, 3) (2, 5), (4, 5), (2, 7), (4, 7), (6, 7)\}$

ILLUSTRATION 12. If R be a relation ≤ in N = { 1, 2, 3,4 , 5, 6,...} such that (a, b) ∈ R, if an only if $a \leq b$, state the nature of the relation.

SOLUTION

The given relation is both

(*i*) reflexive, because $a \leq b$ for every $a \leq$ N, and

(*ii*) transitive, because $a < b$, $b < c \to a < c$

But it is not symmetric because when $3 < 5$, $5 \nless 3$. Further, the given relation is not equivalence as well, because although it is reflexive and transitive, it is not symmetric.

ILLUSTRATION 13. Show that the congruent triangles have equivalence relation.

SOLUTION

A relation is taken to be equivalence when it is reflexive, symmetric and transitive as well. The relation among the congruent triangles is equivalence because.

(i) It is reflexive as xRy, for all x, y ∈ T (a set of traingles)

(ii) It is symmetric as x Ry → yRx for x, $y \in$ T

(iii) It is transitive as xRy, $yRz \to xRz$ for all x, y, $z \in$ T.

ILLUSTRATION 14(a). Find the domain and the range of the following relations.

$R : A \to B$, where $A = \{0, 1, 2, 3, 9\}$, $B = \{-4, 1, 4, 2, 5\}$ and R is "less than or equal to" ($\leq$).

SOLUTION

(i) The domain of the given relation is given by $d(R) = \{0, 1, 2, 3\}$

(ii) The range of the said relation is given by $r(R) = \{1, 4, 2, 5\}$

ILLUSTRATION 14(b). Find the domain and co-domain of the following relation :

$R = \{(x, y) : y = x^2\}$, where $A = \{x : x \in R, B = A\}$

SOLUTION

(i) The domain of the given relation is given by $d(R)$ = a set of all real numbers.

(ii) The co-domain of the given relation is given by $cd(R)$ = a set of all non-negative real numb

17 (b) MEANING, DEFINITION AND CHARACTERISTICS OF FUNCTIO

Meaning

By the term "function", we mean the relationship between any two variables like, supply and p time and distance, volume and freight etc. which are o related with each other that for any value of on them, there corresponds a definite value for the other, and thus the second variable is said to be function of the first one. For example, distance is a function of time or speed, freight is a functio volume of the goods consigned, 4 is the squaring function of 2 etc. A function always explains the na of correspondence between some variables which can be indicated by some formula, graph mathematical equation. It is a special type of relation in which each element of the first set is related to and only one element of the second set.

In other words, it is a subset of some ordered pairs, such that for each elements $x \in A$ there unique element y . B. However, the term 'function' is a Latin word which means some operation. concept of function was introduced in mathematics for the first time by the German Mathemati Leibnitz (1646-1716). It plays an important rule in the field of mathematics basing on which m mathematical devices like, calculus have developed.

Definitions

From the above discussion, the term, 'function' may be defined in various ways as under:

1. "A function is a relationship between two real values, x and y, which are so related corresponding to every value of x we get a finite value of y, whereby y is said to be a function of x".

2. "Given, A and B any two non-empty sets, a function from A to B is a relation in which to e element of A there corresponds a unique element of B."

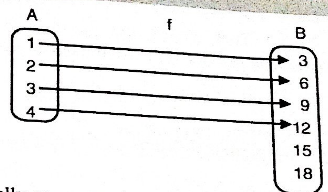
3. "Given, A and B any two non-empty sets, a function from A to B is a subset of A×B such tha ordered pair in f has the same first entry."

4. "A function is a rule, which assigns to every element of the first set A a unique element of second set B".

Characteristics

From the above definitions, the essential characteristics of a function may be analysed as under :

1. It is a logical relation between two variables or sets in which each element of the first se related to only one element of the second set. For example if the two sets A and B are, A = {1, 2, 3, and B = {3, 6, 9, 12, 15, 18}, and we construct another set 'f' there from basing on the rule, *ais treble o* then the resulting set 'f' = {(1, 3), (2, 6), (3, 9), (4, 12)}.

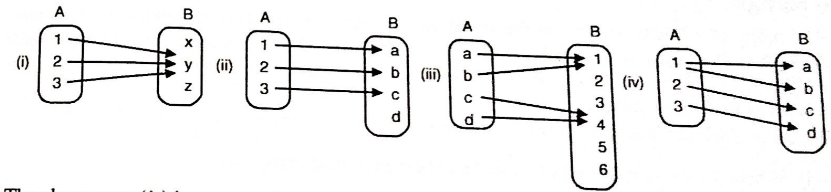


This can be diagrammatically represented as above where we find that 'f' is a correspondence in which every element of A is associated with a unique element of B :

2. Any element(s) of the second set may be related with one or more element of the first set but no element of the first set can have relation with more than one element of the second set. For example, the relation between the wives and husbands, or between the chairs and the students are the case of functional relationship as every wife, ordinarily can have only one husband but a husband can have more than one wife at a time. Similarly, a student ordinarily can sit on one chair only, but a chair can accommodate more than one student at a time.

The following arrow diagrams give clear pictures of functions and non-functions :

Pictures of function



The above case (*iv*) is an example of none function, because the element '1' of the first set. A has correspondence with more than one element of the second set B.

3. Each and every element of the first set must have a correspondence with a unique element of the second set, but it is not necessary that every element of the second set must have correspondence with some element of the first set. This means that some element of the second set may remain unrelated with any element of the first set, but no element of the first set can remain unrelated with any element of the second set.

4. The relationship between the elements of any two sets is based on some rule or logic. This means that there can not be a function between any two objects on any arbitrary correspondence.

5. A function from A to B is a subset of A × B and the domain of the function, f is always equal to A.

6. A function is always denoted by a specific notation which may take any of the following forms :

(*i*) $f(x)$, $F(x)$, $g(x)$, $h(x) : A \to B$ etc.

7. **A function consists of five essential ingredients** *viz.* **: (*ii*) Domain, (*ii*) Co-domain, (*iii*) Range, (*iv*) Image and (*v*) Pre-image.**

These are explained as follows :

(i) Domain

The domain of a function consists of the set of all the first elements (coordinates) of the pairs of a function(s).

Thus, if a function $f = \{(1, 2), (2, 4) (3, 6), (4, 8)\}$, the domain of the function $d(f) = \{1,2,3,4\}$
However, the domain of the algebraic function would be determined as under :

(i) Where, $f(x) = \frac{1}{x}$, $d(f)$ = Set of all real number excluding zero.

(ii) Where, $f(x) = \frac{1}{x-1}$, $d(f)$ = Set of all real numbers excluding 1.

(iii) Where, $f(x)=\sqrt{x}$ and $f(x) \leq 0$, $d(f) = 0 \leq x \leq \infty$.

(iv) Where, $f(x) = \sqrt{4-x}$, $f(x) \geq 0$, $d(f) = -\infty < x < 4$.

(ii) Co-domain

The co-domain of a function is the set of all the elements of the second set B some or all of which may have relation with the elements of the domain of the function. Thus, if A={1,2,3,4},B={2,4,6,8,10} and $f : A \rightarrow B = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$, then the codomain of the function would be set the B *i.e.*, {2, 4, 6, 8, 10}.

(iii) Range

The range of a function refers to the set of all the second elements (coordinates) of the pairs of function. Thus, if a function, $f: A \rightarrow B$ {(1, 3), (2, 6), (3, 9), (4, 12)}, the range of the function would be $r(f) = \{3, 6, 9, 12\}$. When all the elements of co-domain have relation with some elements of the domain the co-domain itself is the range of the function.

In case of algebraic functions, however, the range would be determined as under :

(i) Where $f(x) = \frac{1}{x}$, $r(f)$ = Set of all real numbers excluding zero.

(ii) Where, $f(x) = \frac{1}{x-1}$, $f(f)$ = Set of all real numbers excluding zero.

(iii) Where, $f(x) = \sqrt{x}$, and $f(x) \leq 0$, $r(f) = -\infty < f(x) < 0$.

(iv) Where, $f(x) = \sqrt{4-x}$, and $f(x) \geq 0$, $r(f) = 0 \leq f(x) < +\infty$.

(iv) Image

The term, 'image' refers to the value of the dependent variable y of the pairs (x, y) of a function.
Thus, if $f = \{(1, 2), (3, 4), (5, 6)\}$, then the respective images are 2, 4, and 6.

(v) Pre-image

The term 'pre-image' refers to the value of the independent variable, x of the pair (x, y) of a function. Thus, in the above example, the respective pre-images are 1, 3, and 5.

8. A function of algebraic form any consists of a constant that retains the same value throughout a set of mathematical operations. Conventionally, the initial alphabets like, a, b, c, etc. are used as symbols for constants. There are two types of constants used in a function, *viz.* (*i*) absolute constant, and (*ii*) arbitrary constant.

Absolute constant

An absolute constant is a symbol like π (pie) e (exponential), $\sqrt{2}$, etc. that retains the same value as 3.1429, 2.7183, and 1.414 respectively in all operations and discussions.

Arbitrary constant

An arbitrary constant is a symbol like, a, b, c, etc. that may have any assigned value throughout a set of mathematical operations. The examples of such constants are the radius of a circle, and the sides of a right-angled triangle in forming the trigonometric ratios.

EXERCISE (D)

1. Determine the cardinality of the following sets :

(*i*) A {5, ϕ, (*ii*) A = {7}, (*iii*) {ϕ}, (*iv*) B = {b}, (*v*) A {{a}}, (*vi*) D = {{d} {d}}, (*vii*) A = {{{a}}} (*viii*) E = {0, {ϕ}}

[Ans. [(*i*) 2, (*ii*) 1, (*iii*) 1, (*iv*) 1, (*v*) 1, (*vi*) 3, (*vii*) 1, (*viii*) 2]

2. Fill in the blanks :

(*i*) A void set is denoted by.....
(*ii*) A multition set contains.....elements
(*iii*) A∪B = B .....A (*iv*) A∩B = B ....A
(*iv*) A Δ B= B ....A (*vi*) A–B .....B–A
(*v*) If A ⊂B, then A ∩B = ...
(*vi*) If A ⊂B, then A ∪B = ...
(*vii*) A ∪ϕ =.... (*x*) A∩ϕ = ...
(*viii*) A′∩ B′ = ... (*x*) A ∩ ϕ = ...
(*ix*) A′∩ B′ =....(*x*) (A′)′ = ...
(*xi*) A′∪ B′ =....

[Ans. (*i*) {ϕ}, (*ii*) More than one, (*iii*) ∪, (*iv*) ∩, (*v*) Δ, (*vi*) ≠ (*vii*) A, (*viii*) B, (*ix*) A, (*x*)ϕ, (*xi*) A ∪B)′, (*xii*) A, (*xiii*) (A∩B)′]

3. Fill up the following :

(i) A∪B)∪C = ...
(ii) A ∩ B)∩ C = ...
(iii) A∪(B∩ C) = ...
(iv) {A∩(B∪C) =....
(v) A∪A= ....
(vi) A∩A =....
(vii) A∪U=...
(viii) A∩U= ....
(ix) A′= ......

[Ans. (*i*) A∪(B∪C), (*ii*) C ∩ (B∩C), (*iii*) (A∪B)∩(A∪C), (*iv*) A∩B)∪(A∩C), (*v*)A , (*vi*) A, (*vii*) U, (*viii*) U, (*ix*) U–A]

PROBLEMS

1. A market survey over 1,000 consumers revealed that 730 consumers like the product A, and 455 consumers like the product B. What is the least number that likes both the products ?
Ans. [185]

2. In a survey relating to reading habits of people in a town it was observed that 60% read magazine A, 50% magazine B, 50% magazine C, 30% A and B, 20% B and C, 30% C and A and 10% all the threes. In the fitness of the thing, state (*i*) what percentage read exactly two magazines ? (*ii*) what percentage do not read any of the threes ?
Ans. [(*i*) 50%, (*ii*) 10%]

3. In a survey of 200 students of U.U., the numbers studying various languages were found to be as follows :
English 56, Oriya 60, Hindi 84
English and Oriya 16, English and Hindi 20

Oriya and Hindi 10, and all the three languages 6

In the face of the survey, state through a Venn diagram (*a*) how many students did not study a language and (*b*) how many students had Hindi as their only language.

Ans. [40 and 60]

4. A survey of defect in hardness, finishing and dimensions of 300 items revealed the follow position :

All three defects 15; defect in hardness and finishing 30, defect in dimension and hardness defect in finishing 90, in hardness 69, and in dimension 150. The surveyor was penalys Why ?

Ans. [Then number of items with defect in hardness alone is –6, which is impossible]

5. From a survey of 200 persons, the numbers that subscribe to a certain monthly magazine w observed as follows :

Sept. only 36, Sept. but not Aug. 46

Sept. and July 16, Sept. 52, July 96, July and Aug. 16, None of the threes 48.

Using the set devices find,

(i) How many read Aug. issue ?

(ii) How many read two successive issues ?

(iii) How many read the July issue, if and only if they did not read the Aug. issue ?

And (iv) How many read the Sept. and Aug. issues but not the July issue ?

Ans. [36, 16, 80, none]

6. Using the method of John Venn ascertain the number of students in the commerce faculty of college, if there are three professors, A, B and C and 160 students take tuition from A, 250 fro B, 210 from C, 30 from all the threes, 50 from A and B, 40 from A and C, 80 from B and and 20 do not take tuition from any of them.

Ans. [500]

7. A market research group conducted a survey of 200 consumers and reported that 144 consume like the product A, and 90 the product B. What is the least number of consumers that must ha liked both the products ?

Ans. [34]

8. A survey reveals that 80% of population in Cuttack subscribe for the Prajatantra, and 55% a males. Determine the least possible percentage of males reading the Prajatantra and of female reading the same.

Ans [35%, 25%]

9. In Orissa 60% read the Samaja, 25% the Prajatantra but not the Samaja. Compute by set devic the percentage of those who do not read any. Also, find the highest and the lowest possib figure of those who read the newspaper, the Prajatantra.

Ans. [15%, 85%, 25%]

□□□

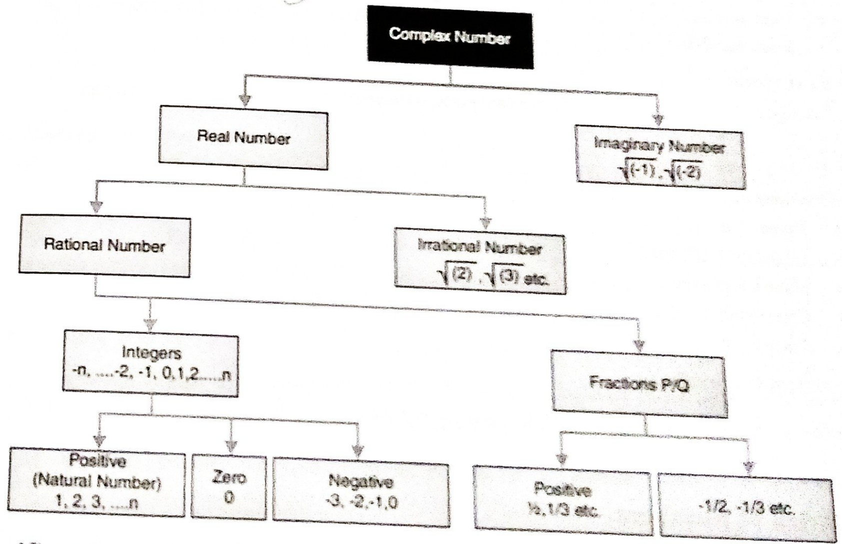
Chapter 2

Number System and Binary Arithmetic

1. INTRODUCTION

By a number system we mean a complete set of unique symbols (characters) and the notes for using the symbols to express any quantity. Numbers refer to numerical figures or constants and they can be expressed in two different forms *viz.* (*i*) as integral quantity (whole numbers without decimal point) and (*ii*) as decimal quantity (numbers with decimal point.)

Number System



Classification of Number System

There may be two types of numbers:

(a) **Real Numbers:** It is the set of all numbers that can be shown on a number line.

For example : 2, –6, 0, $\frac{5}{11}$, $\sqrt{7}$ etc. are all real numbers.

(b) **Complex Numbers:** It is a combination of real and imaginary numbers written as $a + ib$ w
a and b are real numbers and $i = \sqrt{-1}$ is an imaginary number.

Real numbers are further classified into Rational and Irrational Numbers. Different typ
numbers are discussed here as under :

I. Rational Number

These are the numbers which can be expressed in the form of $\frac{a}{b}$ where $b \neq 0$ and a and b are inte

e.g., $\frac{5}{7}, -\frac{11}{12}, \frac{6}{1}, -\frac{1}{2}$ etc. Rational numbers can further be classified into integers and fractions.

(a) Integers

It is a set of all non-fractional numbers lying between $-\infty$ to $+\infty$ denoted by Z or I. For example: {
....., –5, –3, –2, –1, 0, 1, 2, ...+∞}.

Integers can be classified into

- Natural numbers
- Whole number
- Prime numbers
- Odd number
- Even number

(b) Fractions

Fractions are the numbers which have two parts, numerator and denominator, for example:

$\frac{1}{9}, \frac{3}{5}, \frac{5}{13}, \frac{1}{19}$ etc.

Fractions can be classified into

- Proper fractions
- Improper fractions
- Mixed fractions
- Compound fractions
- Complex fractions

II. Irrational Numbers

These are the numbers which cannot be expressed in the form of $\frac{a}{b}$, where $b \neq 0$ e.g. $\sqrt{7}$, $3\sqrt{5}$,...
7.142839, ... etc.

Properties of Numbers

(i) Identity Property

1. **Additive Identity.** Any number + Zero = Number itself, *i.e.* $a + 0 = a$, *e.g.*, $5 + 0 = 5$.
2. **Multiplicative Identity.** Any number × 1 = Number itself, *i.e.*, $a \times 1 = a$, *e.g.*, $7 \times 1 = 7$.

(ii) Inverse Property

1. **Additive Inverse.** If sum of two numbers is zero, then one of the numbers is additive inverse of other number, *i.e.*, $a + (-a) = 0$. Here, $(-a)$ is additive inverse of a.
2. **Multiplication Inverse.** If product of two numbers is 'one', the one number is multiplicative inverse of other, i.e., $a \times \left(\frac{1}{a}\right) = 1$. Here, $\frac{1}{a}$ is the multiplicative inverse of 'a'.

(iii) Associative Property

1. **Associative Property of Addition.** $(a + b) + c = a + (b + c)$

   e.g., $(3 + 5) + 9 = 3 + (5 + 9) = 8 + 9 - 3 + 14 - 17$

2. **Associative Property of Multiplication** $(a \times b) \times c = a \times (b \times c)$

   e.g., $(5 \times 7) \times 11 = 5 \times (7 + 11)$

   $\Rightarrow$ $35 \times 11 = 5 \times 77 = 385$

(iv) Commutative Property

Changing or swapping of numbers is called commutative property.

1. **Commutative Property for Addition**

   $a + b = b + a$

   e.g., $5 + 9 = 9 + 5$

   $\Rightarrow$ $14 = 14$

2. **Commutative Property for Multiplication**

   $a \times b = b \times a$

   e.g., $4 \times 5 = 5 \times 4$

   $\Rightarrow$ $20 = 20$

(v) Distributive Property

This property allows us to remove the bracket from an expression.

$a \times (b + c) = a \times b + a \times c$

e.g., $[10 \times (5 + 8] = [(10 \times 5) + (10 \times 8)]$

$\Rightarrow$ $10 \times 13 = 50 + 80$

$130 = 130.$

(vi) Closure Property

1. **Closure Property for Addition.** Sum of two numbers belongs to the same set as the addent, *e.g.*, $a + b = c$. Here a, b and c belong to the same set of numbers, *e.g.*, (2 + 5) = 7 (all whole numbers)
2. **Closure Property for Multiplication.** Product of two numbers belongs to same set as the factor, *i.e.*, $a \,.\, b = c$, *e.g.*, $3 \times (-7) = (-21)$ (all integers).

(vii) Reflexive Property

According to this property, two numbers are equal to each other only if both the numbers are same, i.e., $a = a$, *e.g.*, $5 = 5$

(viii) Symmetric Property

According to this property, if one number is equal to the second number, then that second numbe[r is] also equal to the first number, *i.e.*, if $a = b$, then $b = a$.

Operation of Numbers

1. Finding a Unit Digit in a Product

EXAMPLE 1. Find the unit's digit in the product

$$108 \times 93 \times 245 \times 51$$

SOLUTION

Product of unit's digits in given numbers

$$8 \times 3 \times 5 \times 1 = 120$$

$\therefore$ Unit digit in the given product is 0.

EXAMPLE 2. Find the unit digit in the product of $(3^{30} \times 6^{24} \times 7^{49})$.

SOLUTION

We know that, unit digit on 3^4 is 1.

$\therefore$ Unit digit in 3^{28} is 1.

Hence, unit digit in $3^{30} = 1 \times 3 \times 3 = 9$

$\therefore$ Unit digit in every power of 6 is 6.

Therefore, unit digit is $6^{24} = 6$

Unit digit in 7^4 is 1.

$\therefore$ Unit digit in 7^{48} is 1.

Hence unit digit in $7^{49} = 1 \times 7 = 7$.

Therefore, product of unit digits in the given number $(3^{30} \times 6^{24} \times 7^{49}) = 9 \times 6 \times 7 = 378$

$\therefore$ Unit digit in the given product = 8.

2. Representation of Rational Numbers

Rational numbers when converted into decimal form can either be a recurring and non-terminating [or] a terminating decimal, e.g., terminating decimal = 3.9, non-terminating and recurring decimal = 3.939393.......

EXAMPLE 3. Convert 8.777.., into a rational number.

SOLUTION

Let $\qquad x = 8.777.....$

Then $\qquad 10x = 87.777....$

$$10x - x = 87.777 .... - 8.777......$$

$$\Rightarrow \qquad 9x = 79$$

$$\Rightarrow \qquad x = \frac{79}{9}$$

This is the rational representation of a non-terminating and recurring decimal number.

3. Remainder Theorem

Dividend = (Divisor $\times$ Quotient) + Remainder

e.g., the division of 5 by 3 gives remainder 2 and quotient 1. Then, $5 = (3 \times 1) + 2$.

EXAMPLE 4. On dividing a certain number by 134, the quotient is 102 and remainder is 10. What is the required number?

SOLUTION

By remainder theorem

$$134 \times 102 + 10 = 13678$$

$\therefore$ Required number = 13678.

Test of Divisibility

Divisibility by 2. A number is divisible by 2, if its last digit (unit digit) is divisible by 2.

Divisibility by 3. A number is divisible by 3, if the sum of its digits is divisible by 3. e.g., 5 + 7 + 8 + 1 = 21.21 is divisible by 3. So, 5781 is divisible by 3.

Divisibility by 4. A Number is divisible by 4, if the number formed by the last two digits is divisible by 4, *e.g.*, 6784.

Here, it is clear that number formed by the last two digits is 84 which is divisible by 4, hence the entire number is divisible by 4.

Divisibility by 5. A number is divisible by 5, if the last digit is either 0 or 5, *e.g.*, 2685, 12970, 38925, ...are divisible by 5.

Divisibility by 6. A number is divisibility by 6, if the number is divisible by both 2 and 3, simultaneously, *e.g.*, 42. 42 is divisible by both 2 and 3, so, it is also divisible by 6.

Divisibility by 7. A number is divisible by 7, when the difference between twice the digit at ones, place and the number formed by the remaining digits is either 0 or a number divisible by 7. For example, 756 is divisible by 7 as $75 - (2 \times 6) = 75 - 12 = 63$ is divisible by 7.

Divisibility by 8. A number is divisible by 8, if the last 3 digits taken together, are divisible by 8. For example, 5144 is divisible by 8, since 144 is divisible by 8.

Divisibility by 9. A number is divisible by 9, if the sum of the digits of the given number is divisible by 9. For example, 4518 is divisible by 9, since the sum of its digits 4 + 5 + 1 + 8 = 18 is divisible by 9.

Divisibility by 10. A number is divisible by 10 when the digit at ones place is 0, e.g., 7250 is divisible by 10.

Divisibility by 11. A number is divisible by 11, if the difference of the sum of its digits at odd places and the sum of its digits at even places, is either 0 or a number divisible by 11. For example, 6105 is divisible by 11, since $(1 + 5) - (6 + 0) = 0$

Number systems and representation of data

There are different types of number systems that are used to represent data. We are already familiar with the decimal number system. The other useful number systems are binary, octal and hexadecimal.

1. Binary Number System

In the binary number system, there are only two choices for representing data–either a "0" or a "1". The base or radix of the binary number system is 2.

2. Octal Number System

In the octal number system, there can be eight possibilities :

"0", "1", "2", "3", "4", "5", "6", "7".

The base or radix of the octal number system is 8.

3. Hexadecimal Number System

In the hexadecimal number system, we have 16 symbols:
"0", "1", "2", "3", "4", "5", "6", "7", "8", "9", "A", "B", "C", "D", "E", and "F".

The base or radix of the hexadecimal number system is 16.

There is a direct correspondence between the binary system and the octal system, with three bina digits corresponding to one octal digit. Likewise, four binary digits translates directly into on hexadecimal digits.

Rules of Number Representation

(i) Commas should not be put anywhere in a number.

(ii) + or — signs may be prefixed to a number. If no sign is prefixed, the number is to be taken as positive one.

(iii) The base or radix of the number should be shown as its suffix. If no base is indicated the numb is to be taken as a decimal number.

(iv) Exponent should be used with a number if it is considered necessary.

(v) There should be 9 significant digits at best, viz. : 1, 2, 3, 4, 5, 6, 7, 8 and 9.

(vi) The sequence of numbers should be presented as a string.

It is very much necessary to have a clear idea about the number systems in order to workout th mathematics and work on the computers. According to the magnitude of the radix (base) of the numbe there are different types of number system, in which a number can be expressed.

The names of certain number system and their base may be cited as follows :

Name of the number system	Base of the system
Decimal system	10
Octal system	8
Hexadecimal system	16
Binary system	2
Weekly system	7 (∵ 1 week = 7 days)
Measurement system	12 (∵ 1' = 12")

Radix (Base)

The total number of digits used in a number system is called its Radix or base. It is always one m than the highest digit of the system.

The digits and their number in the radix of the various number systems are as follows :

(i) Binary system → '0', '1' = 2

(ii) Octal system → 0, 1, 2, 3, 4, 5, 6, 7 = 8

(iii) Decimal system → 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 = 10

(iv) Hexadecimal system → 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F = 16

It may be noted that the base itself cannot be a symbol in a number system.

2. DIFFERENT TYPES OF NUMBER SYSTEM

There are different types of number system depending upon the size of their radix. The most pop among them are : Decimal system, and Binary system.

These are annotated at length in turn as under :

1. DECIMAL NUMBER SYSTEM

This number system is popularly known as the International Number System with which almost all the people of the world are acquainted. It was invented in India. It has led to the rapid development of science and technology all over the world. The famous mathematician Laplace has rightly observed, "It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value, a profound and an important idea which appears so simple to us now that we ignore its true merits."

Under this number system a number is constituted by using any of the ten fundamental digits *viz.* : 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 along with a decimal point and + or — sign. Thus the numbers like 5., 17.5, 2345., — 91.5, —27.23, 879. and 1067. are examples of decimal numbers. Since, this number system is based on 10 fundamental digits the base or Radix of the system is 10 in powers of which any number can be expressed. Thus, the number 75. can be expressed as

$7 \times 10^1 + 5 \times 10^0$ which is equal to 70 + 5 = 75.

Similarly, the number 812.65 can be expressed as

$8 \times 10^2 + 1 \times 10^1 + 2 \times 10^0 + 6 \times 10^{-1} + 5 \times 10^{-2}$.

which is equal to 800 + 10 + 2 + 0.60 + 0.05 = 812.65.

From the above examples, it may be noted that under the decimal number system the value of a number depends upon the following four factors :

(i) face value of the digits of which it is composed,

(ii) place value of such digits with reference to their distance from the decimal point,

(iii) Base or Radix of the system, and

(iv) algebraic sign *i.e.* + or —.

Face Value

In the number 12345, each digit starting from the right has its face value like 5, 4, 3, 2 and 1 and place value like 5 ones, 4 tens, 3 hundreds, 2 thousands and 1 billion respectively. The place value of a digit depends on its position in the number with reference to the position of the decimal point therein. In the absence of any specific information the base of the number is to be taken as 10, its algebraic sign as + and the number itself as a decimal number.

Thus, the above number 12345. can be expressed in powers of its base 10 as follows :

5 ones	$= 5 \times 10^0 =$	5
4 tens	$= 4 \times 10^1 =$	40
3 hundreds	$= 3 \times 10^2 =$	300
2 thousands	$= 2 \times 10^3 =$	2000
1 billion	$= 1 \times 10^4 =$	10000
Total of the number	=	12345

From the above example the Base or Radix of a decimal number may be defined as the number which raised to 0 power gives the lowest value, raised to the 1st power gives the second place value, raised to the 2nd power gives the third place value and so on.

Thus,

10^0 = 1., the lowest place value

10^1 = 10., the second place value

10^2 = 100., the third place value

10^3 = 1000., the fourth place value

10^n = $(n + 1)$th place value.

It may be noted that the base itself cannot be a fundamental digit in a number because, such a number system consists of only 10 fundamental digits *viz.* : 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 and the base 10 itself cannot have a place therein.

Place Value

In a decimal number, the place value of a digit starting from the right is 10 times the place value of the digit to its right. Conversely, the place value of a digit starting from the left is 1/10th of the place value of the digit to its left.

Thus, in the number 12345, the place value of the different digits may be cited as follows :

Digit	Place value
5	ones (*i.e.* 10^0)
4	tens (*i.e.* 10^1)
3	hundreds (*i.e.* 10^2)
2	thousands (*i.e.* 10^3)
1	billion (*i.e.* 10^4)

Thus a decimal whole number can be represented by the model

$$N = d^{n+1}b^n + d_n b^{n-1} + d_{n-1} b^{2-2} + d_3 b^2 + d_2 b^1 + d_1 b^0$$

where, N = a decimal whole number

d = a digit of the number

d_{n+1} = digit in the $(n+1)$th position starting from the right

b = base of the number *i.e.* 10

n = exponent of the base for $(n+1)$th digit position.

Different Types of Decimal Numbers

Decimal numbers, can again be classified into three types, namely,

(*i*) Integral Numbers

(*ii*) Fractional Numbers

and (*iii*) Mixed or Real Numbers.

These are discussed as below.

(*i*) **Integral Numbers.** A set of digits including zeroes having no fractional part or digits to the right of its decimal point is called an integral or whole number. It may be either positive or negative in nature. The positive sign may not be put but the negative sign must be prefixed to it.

Thus, 1234., —345., 5078., 500., —35., 0., and 9. are examples of integral numbers.

Characteristics

The chief characteristics of an integral number may be outlined as under :

(*i*) It does not have any digit in its decimal part.

(*ii*) It may or may not be preceded by a + sign when it is positive.

(*iii*) It must be preceded by a —sign when it is negative.

(*iv*) It is closed with regard to addition, subtraction and multiplication. This means that sum, product of any two or more integral numbers results in another integral number.

(*v*) The absolute value of two or more numerically equal integers is the same notwithstanding their plus or minus signs. Thus, the absolute value of 5 and —5 is same and is equal to 5 irrespective of their + and — signs. This absolute value of an integer n is denoted by the signum n or $|n|$

$|n| = n$, where $n \geq 0$

$|n| = -n$, where $n < 0$

(*vi*) It is called a natural number if it is more than zero *i.e.* $n > 0$.

(*vii*) The left most digit of an integer has the highest place value and is called the Most Significant Digit (MSD). On the other hand, the right most digit of an integer has the lowest place value and is called the Least Significant Digit (LSD).

(*ii*) **Fractional Numbers.** A set of digits including zeroes to the right of a decimal point having no integral part or digits to the left of its decimal point is called a fractional number. Like an integer it may be either a positive or a negative value. .756, —.35 —.3457,—.123 are examples of fractional numbers. The place value of a digit in such a number is 1/10th of the place value of the digit to its left and the place value of each succeeding digit goes on decreasing in order of 10^{-1}, 10^{-2}, 10^{-3} and so on. Thus in the fractional number .765, the place value the different digits may be cited as follows :

Digit	Place value
7	1/10th of the unit *i.e.* $7 \times 10^{-1} = 0.70$
6	1/100th of the unit *i.e.* $6 \times 10^{-2} = 0.06$
5	1/1000th of the unit *i.e.* $5 \times 10^{-3} = 0.005$

Thus, the fractional number .765

$$= 7 \times 10^{-1} + 6 \times 10^{-2} + 5 \times 10^{-3}$$
$$= 0.70 + 0.06 + 0.005$$
$$= 0.765$$

As such a fractional number can be expressed by the following model,

$$F = d_1 b^{-1} + d_2 b^{-2} + d_3 b^{-3} + \ldots + d_n b^{-n}$$

where, F = a fractional number

d = a digit in the fractional number

b = base of the system *i.e.* 10 and

n = the exponent of the base for the digit in the nth position when the digit immediately after the decimal point is the digit in the first position.

(**iii**) **Mixed or Real Numbers.** A decimal number that contains both the integral and fractional parts is called a mixed or a real number. It may be either a positive or a negative value like, 37.594, —25.32, 375968.35, 4.8695, —17.486, and 25.0. The part of the number to the left of its decimal point is called integral part and the part of the number to the right of its decimal point is called the fractional part.

Thus, the integral and the fractional parts of the above cited numbers may be distinctly shown as under :

Real Number	Integral Part	Fractional Part
37.594	37.00	0.594
—25.32	—25.00	—0.32
375968.35	375968.00	0.35
4.8695	4.00	0.8695
—17.486	—17.00	—0.486
25.0	25.00	0.0

From the above examples of real numbers it may be noted that the pr... point in a number characterises it as a real, fractional,or an integral numbe... integer whereas the number 25.0 is a real number although both of them... magnitude.

The place value of each digit in a mixed number is determined in the same manner as exp... the integral and fractional numbers.

Thus, the place values of the different digits in the number, 37.594 will be as follows :

$$= 3 \times 10^1 + 7 \times 10^0 + 5 \times 10^{-1} + 9 \times 10^{-2} + 4 \times 10^{-3}$$
$$= 30 + 7 + .5 + .09 + .004 = 37.594.$$

As such a mixed or real number can be expressed by the following model,

$$M = d_{n+1} b^n + d^n b^{n-1} + d^{n-1} b^{n-2} + ... + d^3 b^2 + d^2 b^1 +$$
$$d_1 b^0 + d_1 b^{-1} + d_2 b^{-2} + d_3 b^{-3} + \ldots + d_n b^{-n}.$$

where, M = a mixed decimal number,

and all other symbols carry the same meaning as explained earlier.

Form of representation

There are two forms of representation of a real number. They are :

(*i*) Fixed point representation form and

(*ii*) Floating point or Exponential representation form.

These are explained as below :

(i) Fixed Point Representation Form. In this form of representation a real number is written... detail with the decimal point being fixed in the correct position between the digits.

Thus, 125.48, —3.25, 0.3789, 0.0000789 etc. are examples of the fixed point representation of r... numbers. This form of representation is convenient when the number of significant digits in a number li... 25.35, 12560.75 etc. is not very large. But if the number of digits in a number like 123.0000005... 0.0000000035 etc. is very large, the fixed point form of notation is considered inconvenient for, in th... case it will consume both more space and more time. However, in this form of representation the ratio... representations like 3/5, $5\frac{2}{3}$, $4\frac{3}{7}$ etc. are not permitted.

(ii) Floating Point or Exponential Representation Form. This form of representation of a re... number is considered more convenient when its value is either very large like 98765432100000000.00... very small like 0.00000001234567.

In this form of representation a real number is expressed as a combination of mantissa and expo... by shifting its decimal point by some places either to the right or to the left according to the requirem... and using the required exponent to the base along with the reduced number. Thus, in this form the gi... number is represented shortly in three parts *viz. : mantissa, base and exponent*. The form of this notati... runs as under :

$N = \pm d_1 d_2 d_3 \ldots \times 10^{\pm n}$ d_1, d_2, d_3 etc. = digits of the real number

N = the given number   10 = the base or radix

n = the exponent.

Thus, a real number 5437.65 may be represented in any of the following floating point forms :

543.765×10^1, or 0.543765×10^4, or

54.3765×10^2, or 54376.5×10^{-1}, or

5.43765×10^3, or 543765×10^{-2}.

...d that when the decimal point is to be shifted by one place to its left the exponent is to ...e and when the decimal point is to be shifted by one place to its right, the exponent is to ...e.

...oint representation form is again of two types, *viz.* :

...form

(*i*) Popular form.

...) Standard Floating Point Form. This form is also otherwise called as the normalized exponential ...m. In this form the mantissa is made to lie between the value 0.1 and 1. Thus the standard foating point form of the number 437.65 will be : $.43765 \times 10^3$.

This form of notation is represented by $m \times b^n$

where m = mantissa,

b = base,

n = exponent.

and $1/b \leq m$ and $m \leq 1$.

The above process of shifting the mantissa to the left till its most significant digit is non zero is called Normalization. This is done to preserve the maximum number of useful digits.

(*ii*) Popular Floating Point Form. In this form the mantissa is made to lie between the value 1 and 10. In other words, in this form of representation, the first significant digit of the number only is retained in the unit place and all other digits are shown in the decimal place. Thus, the popular floating point form of the number 437.65 will be 4.3765×10^2 and that of .0009875 will be 9.875×10^{-4}.

ILLUSTRATION 1. Show the difference between the fixed and floating point representation of the following decimal numbers :

$\boxed{1234.56789;\ 23.0534;\ 0.0005734;\ 9897.0321\ ;\ \text{and}\ 45.32109.}$

SOLUTION

Difference Between the Fixed and Floating Point Representation

Fixed point representation	Floating point representation	
	Standard form	Popular form
1234.56789	0.123456789×10^4	1.23456789×10^3
23.0534	0.230534×10^2	2.30534×10^1
0.0005743	0.5743×10^{-4}	5.743×10^{-5}
9897.0321	0.98970321×10^4	9.8970321×10^3
45.32109	0.4532109×10^2	4.532109×10^1

ILLUSTRATION 2. State with reasons which of the following numbers are in the standard floating point form. Also state their standard form if they are otherwise.

$$17.38 \times 10^4\ ;\ 10^6;\ 0.9876 \times 10^{-25};\ 1.45 \times 10^{13}\ ;$$
$$0.6789 \times 10^{-4};\ 0.001564 \times 10^5\ ;\ 23.57 \times 10^2$$

SOLUTION

(*i*) 17.38×10^4	This is not in a standard floating point form because, here the mantissa 17.38 does not lie between the value 0.1 and 1. The standard floating point form of this number would be 0.1738×10^6.
(*ii*) 10^6	This is not in the form of a floating point number because there is no mantissa

	in the given number. Hence, the question of standard floating point form does not arise.
(iii) 0.9876×10^5	This is very much in the standard floating point form because, the mantissa .9876 lies between the value 0.1 and 1.
(iv) 1.45×10^{13}	This is not in a standard floating point form because, the mantissa 1.45 exceeds the value 1. However, it is in the popular form as the mantissa lies between the value 1 and 10. The standard form of this number would be 0.145 × 1014.
(v) 0.6789×10^{-4}	This is quite in the standard floating point form because, the mantissa .6789 lies between 0.1 and 1.
(vi) 0.001564×10^5	This is not in the standard form because, the mantissa 0.001564 is less than 0.1. This standard form of the number would be 0.1564 × 103.
(vii) 23.57×10^2	This is not in the standard form because, the mantissa 23.57 is more than 1. Its standard form would be 0.2357 × 104.

ILLUSTRATION 3. Present the following fixed point real numbers in different floating point notations :

5678.9; 1234000000000.0; 0.00000009876; 0.00345; —10.0012 and 0.

SOLUTION

Representation of the Fixed Point Real Numbers in Different Forms of Floating Point Notations

Fixed point numbers	Standard floating point notation	Popular floating point notation
5678.9	0.56789×10^4	5.6789×10^3
1234000000000.0	0.1234×10^{13}	1.234×10^{12}
0.00000009876	0.9876×10^{-7}	9.876×10^{-8}
0.00345	0.345×10^{-2}	3.45×10^{-3}
—10.0012	-0.10×10^0	-1.0×10^1

Note. 1. *The extreme significant digits as in the case of* —10.0012, *viz* : 12 *have been ignored as their value is very insignificant.*

2. *It is not possible to represent '0' in any floating point form because it is less than* 0.1 *and it will give zero when multiplied by any number.*

EXERCISE (A)

1. *What do you mean by the Number System ? State the rules of number representation.*
2. *Enumerate the different types of number system and explain the concept of Radix.*
3. *What do you mean by decimal number system ? Explain with illustrations the various types of decimal numbers.*
4. *Explain with numerical examples the various forms of floating point representation of a real number.*
5. *Outline the chief characteristics of an integral number.*
6. *Write short notes on :*
   (a) *Radix* (b) *Normalisation*
   (c) *Exponential form of representation.*

7. Write short notes on :
   (i) Fractional numbers (ii) Mixed number
   (iii) Floating point representation.
8. Show the difference between the fixed and floating point representation of the following decimal numbers.

   34567.4321, 15.3453, 0.0001234, 123.0321, 3734.12345
9. State with reasons which of the following numbers are in the standard floating point form. Also, state their standard form, if they are otherwise.

   25.75×10^5, 0.1234×10^{-15}, 1.47×10^{11}, 0.12345×10^{-3}, 0.007854×10^3, 71437×10^2
10. Present the following fixed point real numbers in different floating point notations :

   9876.5, 5678900000.0, 0., 0.0000012345, 0.000123, —10.000372
11. (a) Convert the following into their fractional form :
   (*i*) 1.6666 .... (*ii*) 1.222 .... **[Ans.** 5/3, 11/9]

   (b) Convert the following into their decimal form :
   (*i*) $\sqrt{5}$ (*ii*) 1/7 (*iii*) 5/23

2. BINARY NUMBER SYSTEM

This number system is very useful for working of the computers because a computer can understand no other system except the binary one. Under this number system a number is constituted with any of the two fundamental digits viz. : 0 and 1 which are frequently referred to as binary digits. The examples of such numbers are 1010, 1111., 0000, 01.01., 1100.10 etc. Since this number system is based on 2 basic digits, 0 and 1 the radix or base of this system is 2. The place value of each digit in a binary number is twice the place value of the digit to its right. Such numbers are usually written with the radix or base 2 indicated as its suffix. Thus, the above binary numbers will be presented as : $(1010)_2$, $(1111.)_2$, $(01.01)_2$, $(1100.10)_2$.

The chief points of difference between a decimal number and a binary number may be outlined as follows :

Point of difference	Decimal numbers	Binary numbers
1. Base	10	2
2. Basic digits	0, 1, 2, 3, 4, 5,, 6, 7, 8, 9	0, 1
3. Place value of a digit	10 times of the digit to its right	2 times of the digit to its right
4. Indication of radix	not necessary	necessary *i.e.* 2
5. Examples	(125), (100), (25.3504)	$(0101)_2$, $(1111)_2$, $(0000)_2$

Special terms

The special terms used in connection with the binary number system are explained as follows :

Bit. The term, bit is an abbreviation of the words Binary digit. It takes the form of either of the binary digits 0 and 1 to represent exactly one character.

Binary digit. It is a logical 0 or 1 that represents a passive or an active state respectively of a component in an electrical circuit.

Byte. It is a group of 8 bits. It is the smallest unit which can represent a data item or a character.

Nibble. It is a group of 4 bits. It is very often used in the binary system for each number.

Advantage

As pointed out earlier the binary system is suitable for computers because it consists of only two digits 0 and 1 and a computer works on electrical and electronic devices which can exist in and understand only two states *i.e. inactive or off state (0) and active or on state (1).*

Disadvantage

Inspite of the above valuable advantage, the binary system suffers from the defect of expansion. This means that a number which can be expressed with less number of digits in any other system will need a large number of digits to be expressed under the binary system. For example, the number 5 needs only one digit i.e. 5 to be expressed in the decimal system, octal system and hexadecimal system as well. But it will need three digits i.e. 101 to be expressed in the binary system. (Its computation procedure is detailed in the conversion section discussed little later).

3. CONVERSION OF NUMBER SYSTEM

The various number system are quite capable of being converted into each other to meet the needs of a situation. A decimal number can be directly converted into a number of any other system. Similarly, a number of any other system, also, can be very easily converted straightway to a decimal number. But the conversion of a number of any other system like binary, octal and hexadecimal to a number of any such system involves little difficulties. However, the process of conversion of the decimal numbers into the binary numbers and vice versa is explained at length as below :

CONVERSION OF DECIMAL NUMBERS INTO BINARY NUMBERS

There are two different methods of converting a decimal number into a binary number. They are :

1. *Remainder method or the method of division.*
2. *Power method or the method of subtraction.*

These methods are detailed as below :

(i) Remainder method or the method of division

Under this method to convert a decimal number to a binary number, the following steps are to be taken up in turn :

(i) Divide the given decimal number by the radix of the binary system to which the number is to be converted and places the quotient Q_1 and the remainder R_1 (including zero) distinctly in a horizontal line below the number.

(ii) Divide again the quotient Q_1 by the same radix and place the quotient Q_2 and the remainder R_2 distinctly in a horizontal line below the quotient Q_1.

(iii) Continue this process of division till the quotient is reduced to less than the radix.

(iv) Compile all the remainders including that of the last quotient in a reverse order as Q_n, R_n, R_{n-1}, R_{n-2}, Rn_3, Rn_2, Rn_1, and get the desired equivalent number.

The following illustrations will show how the decimal numbers can be converted into binary numbers under the above method.

ILLUSTRATION 4. Convert the following decimal numbers into their binary equivalents using the *remainder method :*

7., 28., 940. and 5731.

SOLUTION

Conversion of the Decimal Numbers to Their Binary Equivalents by the Remainder Method

Radix of the binary system = 2

Hence, the conversion process will run as under :

(i) 7

Radix of the number	Decimal number	Remainder
2	7	
2	3	1 (R_1)
	(Q_n) (1 radix 2)	1 (R_2)

Compiling all the remainders including the last quotient in the reverse order we get, 111. Thus the binary equivalent of the decimal number 7 = $(111)_2$.

(ii) 28

Radix	Dec. No.	Remainder
2	28	
2	14	0 (R_1)
2	7	0 (R_2)
	3	1 (R_3)
2	(Q_n) (1 radix 2)	1 (R_4)

Compiling all the remainders including that of the quotient in the reverse order we get, 11100.
Thus, the binary equivalent of the decimal number 28 = $(11100)_2$.

(iii) 940

Radix	Dec. No.	Remainder
2	940	
2	470	0 (R_1)
2	235	0 (R_2)
2	117	1 (R_3)
2	58	1 (R_4)
2	29	0 (R_5)
2	14	1 (R_6)
2	7	0 (R_7)
2	3	1 (R_8)
2	(Q_n) (1 radix 2)	1 (R_9)

Compiling all the remainders including that of the quotient in the reverse order we get 1110101100.
Thus the binary equivalent of the decimal number 940 = $(1110101100)_2$

(iv) 5731

Radix	Dec. No.	Remainder
2	5731	
2	2865	1
2	1432	1

2	716	0
2	358	0
2	179	0
2	89	1
2	44	1
2	22	0
2	11	0
2	5	1
2	2	1
2	(Q_n) (1 radix 2)	0

Compiling all the remainders including that of the quotient in the reverse order we get 1011001100011.

Thus, the binary equivalent of the decimal number

$5731 = (1011001100011)_2$.

(ii) Power Method or The Method of Subtraction

Under this method, to convert a decimal number to a number of binary system the following steps need to be taken up in turn.

(i) Find the different values upto the value of the given decimal number by raising the radix of the binary system to the powers permissible for the purpose.

(ii) Subtract the highest of the values thus obtained or its multiple from the given number.

(iii) Subtract the next deductible higher value or its multiple from the remainder.

(iv) Repeat the above process of subtraction till the remainder is reduced to zero.

(v) Represent the subtracted power values by their repsective multiplication factor and the

(vi) unsubtracted power values by zeroes.

(vii) Compile the above representing values in a descending order and get the desired equivalent.

The following illustrations would show how a decimal number can be converted by the power method.

ILLUSTRATION 5. Using the power method of conversion obtain the binary equivalents of the following decimal numbers :

(i) 28, and (ii) 940.

SOLUTION

Radix of the binary system = 2.

(i) For 28

The different power values of the radix 2 that fall within the given number 28 are :

	1,	2,	4,	8	and	16
i.e.	(2^0)	(2^1)	(2^2)	(2^3)		(2^4)

arrying the process of subtraction we have :

Dec. Number and Subtrahenal		Multiplication Factor of Subtracted Values
ax. value)	$= 2^4 \times 1 \rightarrow$	1

-8 (next max. value) 4	$= 2^3 \times 1 \rightarrow$	1
-4 (next max. value) 0	$= 2^2 \times 1 \rightarrow$	1

Representing the subtracted power values by their respective multiplication factors and the unsubtracted power values by zeroes in a descending order we have,

Power values	:	16	8	4	2	1
Representing values	:	1	1	1	0	0

Compiling the above representing values we get the binary equivalent of the number 28 = $(11100)_2$.

(ii) For 940

The different power values of the radix 2 that fall within the given number 940 are :

	1	2	4	8	16	32	64	128	256	512
i.e.	(2^0)	(2^1)	(2^2)	(2^3)	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)	(2^9)

Carrying the process of subtraction we have :

Dec. Number and Subtrahenal			Multiplication Factor of Subtracted Values
	940		
—Max. value	512 / 428	$= 2^9 \times 1 \rightarrow$	1
—next max. value	256 / 172	$= 2^8 \times 1 \rightarrow$	1
—next max. value	128 / 44	$= 2^7 \times 1 \rightarrow$	1
—next max. value	32 / 12	$= 2^5 \times 1 \rightarrow$	1
—next max. value	8 / 4	$= 2^3 \times 1 \rightarrow$	1
—next max. value	4 / 0	$= 2^2 \times 1 \rightarrow$	1

Representing the subtracted power values by their respective multiplication factors and the remaining power values by zeroes in a descending order we have :

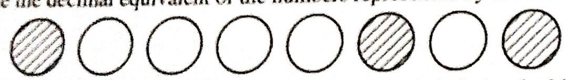
Power values :	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Representing values :	1	1	1	0	1	0	1	1	0	0

Compiling the above representing values we get the binary equivalent of the number 940 = $(1110101100)_2$.

EXERCISE (B)

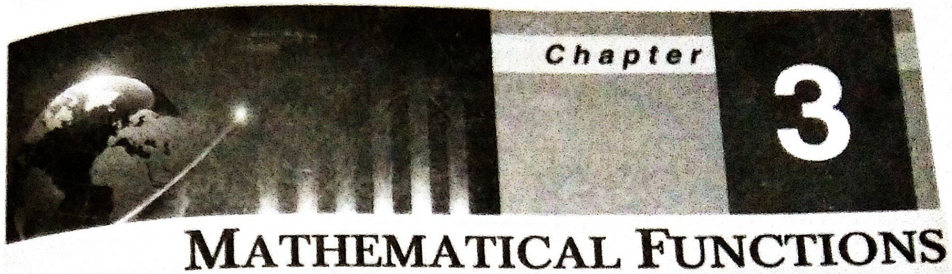
1. What do you mean by the Binary number system ? Discuss its relative advantages and disadvantages.
2. Distinguish between decimal number system and binary number system.

3. Define and explain the following terms in connection with a binary number.
   (i) Bit (ii) Byte
   (iii) Nibble (iv) Binary digit
4. What do you mean by Radix and Radix conversion ? Explain the procedure of converting a binary number into a decimal number.
5. What do you mean by conversion of a number system ? Explain in brief the methods of converting a decimal number into a binary number.
6. Explain the following methods of conversion
   (i) Summation method (ii) Subtraction method
   (iii) Multiplication method (iv) Division method
   (v) Double Babble method
7. (a) 'Computers work on binary number system'. Comment.
   (b) How many binary digits are there ? Which symbols are used for, thereon, and what are they usually called ?
8. Explain the algorithm of the following methods of conversion.
   (i) Remainder method of conversion
   (ii) Power method of conversion
   (iii) Method of converting a decimal fraction into a binary fraction.
9. **Fill in the blanks in the following :**
   (i) There are .... characters in the decimal system.
   (ii) A byte consists of .... bits.
   (iii) There are 16 characters in .... system.
   (iv) A byte contains .... nibble.
   (v) A nibble consists of .... bits.
   (vi) The base of a number system is equal to .... digits used in its.
   (vii) Binary numbers need more place for representation because, their .... is small.
10. (a) Indicate the radix of the following numbers :
    4,57, 258 A9B8H 11010.111
    (b) Indicate the decimal equivalent of the numbers represented by the following circles:



where the shaded circles represent the bit, 1 and the unshaded circles the bit, 0.
11. Convert the following binary numbers into their decimal equivalents.
    (i) 11011010 (ii) 10010011001
    (iii) 1111000111 (iv) 101010101
    (v) 100100101001 (vi) 101100110
12. Convert the following decimal numbers into their binary equivalents :
    (i) 38.205 (ii) 0.8156
    (iii) 22.8125

□□□



MATHEMATICAL FUNCTIONS

1. MEANING, DEFINITIONS AND CHARACTERISTICS OF A FUNCTION

Meaning

By the term 'function', we mean the relationship between any two variables like, supply and price, time and distance, volume and freight etc. which are so related with each other that for any value of one of them, there corresponds a definite value for the other, and thus the second variable is said to be the function of the first one. For example, distance is a function of time or speed, freight is a function of volume of the goods consigned, 4 is the squaring function of 2 etc. A function always explains the nature of correspondence between some variables which can be indicated by some formula, graph or mathematical equation. **It is a special type of relation in which each element of the first set is related to one and only one element of the second set.**

In other words, *it is a subset of some ordered pairs, such that for each element* $x \in$ A *there is a unique element* $y \in$ B. However, the term 'function' is a Latin word which means some operation. The concept of function was introduced in mathematics for the first time by the German Mathematician, **Leibniz** (1646-1716). It plays an important role in the field of mathematics basing on which many mathematical devices like, calculus have developed.

Definitions

From the above discussion, the term, 'function' may be defined in various ways as under :

1. "A function is a relationship between two real values, x and y, which are so related that corresponding to every value of x we get a finite value of y, whereby y is said to be a function of x".

2. "Given, A and B any two non-empty sets, a function from A to B is a relation in which to every element of A there corresponds a unique element of B."

3. "Given, A and B any two non-empty sets, a function from A to B is a subset of A × B such that no ordered pair in f has the same first entry."

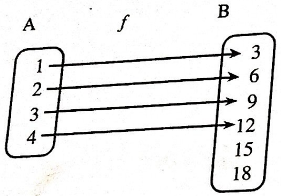
4. "A function is a rule, which assigns to every element of the first set A a unique element of the second set B".

Characteristics

From the above definitions, the essential characteristics of a function may be analysed as under :

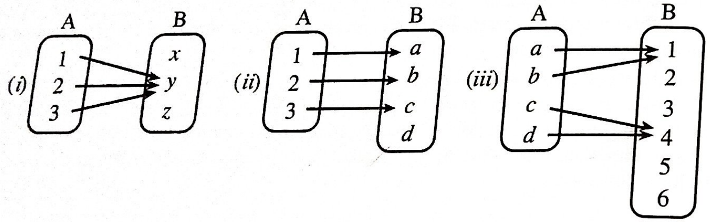
1. It is a logical relation between two variables or sets in which each element of the first set is related to only one element of the second set. For example, if the two sets A and B are, A= {1, 2, 3, 4}

and B = {3, 6, 9, 12, 15, 18}, and we construct another set 'f' therefrom basing on the rule, a is treble of b then the resulting set 'f' = {(1, 3), (2, 6), (3, 9), (4, 12)}. This can be diagrammatically represented as under where we find that 'f' is a correspondence in which every element of A is associated with a unique element of B.

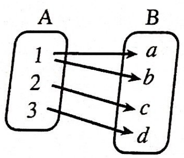


2. Any element(s) of the second set may be related with one or more element of the first set but no element of the first set can have relation with more than one element of the second set. For example, the relation between the wives and husbands, or between the chairs and the students are the case of functional relationship as every wife, ordinarily can have only one husband but a husband can have more than one wife at a time. Similarly, a student ordinarily can sit on one chair only, but a chair can accommodate more than one student at a time. The following arrow diagrams give clear pictures of functions and non-functions :

Pictures of function



Pictures of non-function



The above case is an example of none function, because the element '1' of the first set A has correspondence with more than one element of the second set B.

3. Each and every element of the first set must have a correspondence with a unique element of the second set, but it is not necessary that every element of the second set must have correspondence with some element of the first set. This means that some elements of the second set may remain unrelated with any element of the first set, but no element of the first set can remain unrelated with any element of the second set.

4. The relationship between the elements of any two sets is based on some rule or logic. This means that there can not be a function between any two objects on any arbitrary correspondence.

5. A function from A to B is a subset of A × B and the domain of the function, f is always equal to A.

6. A function is always denoted by a specific notation which may take any of the following forms :

(*i*) $f(x)$, $F(x)$, $g(x)$, $h(x)$, $f: A \rightarrow B$ etc.

7. A function consists of five essential ingredients *viz.* : (*i*) Domain, (*ii*) Co-domain, (*iii*) Range, (*iv*) Image and (*v*) Pre-image.

These are explained as follows :

(*i*) Domain

The domain of a function consists of the set of all the first elements (coordinates) of the pairs of a function(s). Thus, if a function

$$f = \{(1, 2), (2, 4) (3, 6), (4, 8)\}$$

then the domain of the function would be,

$$d(f) = \{1, 2, 3, 4\}$$

However, the domain of the algebraic function would be determined as under :

(*i*) Where, $f(x) = \frac{1}{x}$, $d(f)$ = Set of all real numbers excluding zero.

(*ii*) Where, $f(x) = \frac{1}{x-1}$, $d(f)$ = Set of all real numbers excluding 1.

(*iii*) Where, $f(x) = \sqrt{x}$ and $f(x) \leq 0$, $d(f) = 0 \leq x \leq \infty$.

(*iv*) Where, $f(x) = \sqrt{4x}$, and $f(x) \geq 0$, $d(f) = -\infty < x < 4$.

(*ii*) Co-domain

The co-domain of a function is the set of all the elements of the second set B some or all of which may have relation with the elements of the domain of the function. Thus, if A = {1, 2, 3, 4}, B = {2, 4, 6, 8, 10} and $f: A \rightarrow B$ = {(1, 2), (2, 4), (3, 6), (4, 8)}, then the codomain of the function would be set the B *i.e.*, {2, 4, 6, 8, 10}.

(*iii*) Range

The range of a function refers to the set of all the second elements (coordinates) of the pairs of a function. Thus, if a function, $f: A \rightarrow B$ = {(1, 3), (2, 6), (3, 9), (4, 12)}, the range of the function would be, $r(f)$ = {3, 6, 9, 12}. When all the elements of a co-domain have relation with some elements of the domain, the co-domain itself is the range of the function.

(*iv*) Image

The term, 'image' refers to the value of the dependent variable y of the pairs (x, y) of a function : Thus, if f = {(1, 2), (3, 4), (5, 6)}, then the respective images are, 2, 4, and 6.

(*v*) Pre-image

The term 'pre-image' refers to the value of the independent variable, x of the pair (x, y) of a function. Thus, in the above example, the respective pre-images are 1, 3, and 5.

8. A function of algebraic form may consists of a constant that retains the same value throughout a set of mathematical operations. Conventionally, the initial alphabets like, a, b, c, etc. are used as symbols

for constants. There are two types of constants used in a function, viz. (i) absolute constant, and (ii) arbitrary constant.

(i) Absolute constant

An absolute constant is a symbol like π (pie), e (exponential), $\sqrt{2}$, etc. that retains the same value as 3·1429, 2·7183, and 1·414 respectively in all operations and discussions.

(ii) Arbitrary constant

An arbitrary constant is a symbol like, a, b, c, etc. that may have any assigned value throughout a set of mathematical operations. The examples of such constants are the radius of a circle, and the sides of a right-angled triangle in forming the trigonometric ratios.

2. DIFFERENT TYPES OF FUNCTION

There are different types of functions on two different bases *viz* : (*i*) theory of sets and (*ii*) theory of number system. They are briefly identified here as under :

I. On the basis of theory of sets

1. Into function

A function is said to be an into function when there remains atleast one element of the co-domain which has no relation with any element of the domain. In other words, a function is considered as an into function when its range is a proper subset of its co-domain i.e. r(f) $\subset$ cd (f).

Examples :

(*i*) When A $= \{1, 2, 3\}$, B $= \{0, 1, 2, 3, 4\}$, and f : A $\to$ B $= \{(1, 2), (2, 3), (3, 4)\}$

(*ii*) When A $= \{a, b, c, d\}$, B $= \{a, e, i, o, u\}$, and

f : A $\to$ B $= \{(a, a), (b, e), (c, i), (d, o)\}$

2. Onto function or Surjective function

A function is said to be an onto function when all the elements of the co-domain of the function are related to some elements of the domain of the function. In other words, a function is considered an onto function when its range is equal to its co-domain i.e., r(f) = cd (f). Such functions are also otherwise called *Surjective functions.*

Examples :

(*i*) When A $= \{1, 2, 3\}$, B $= \{a, b, c\}$, and $f = \{(1, a), (2, b), (3, b)\}$

(*ii*) When X $= \{1, 2, 3, 4, 5\}$, Y $= \{a, b, c, d\}$, and

$f = \{(1, a), (2, b), (3, c), (4, d), (5, d)\}$

3. One one (1-1) function or injective function

A function is said to be a one-one function when no element of its co-domain is related with more than one element of its domain. Such functions are also called Injective functions.

Examples :

(*i*) A $= \{1, 2, 3\}$, B $= \{3, 6, 9, 12\}$, and $f := \{(1, 3), (2, 6), (3, 9)\}$

(*ii*) X $= \{1, 3, 5, 7\}$, Y $= \{1, 9, 25, 49\}$, and

$f = \{(1, 1), (3, 9), (5, 25), (7, 49)\}$

4. Many-one function

A function is said to be a many-one function when two or more elements of its domain are related to one element of its co-domain. In other words, a function is considered Many-one function where two or more pre-images of the domain have the same image in its co-domain.

Examples :

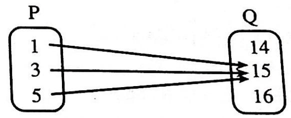
(*i*) Where, A $= \{a, e, i, o, u\}$, B $= \{p, q, r, s\}$ and

f : A$\to$ B $= \{(a, p), (e, q), (i, r), (o, s), (u, s)\}$

(*ii*) Where, P $= \{1, 3, 5\}$, Q $= \{14, 15, 16\}$, and

$f = \{(1, 15), (3, 15), (5, 15)\}$

The following arrow diagram gives the picture of such a function :



5. One-one into function

A function is said to be a one-one into function when it satisfies the requirements of both the one-one and into functions as under :

(*i*) each element of the domain is associated with a separate element of the co-domain.

(*ii*) there remains atleast one element in the co-domain which is not associated with any element of the domain.

Examples :

(*i*) Where, A $= \{a, b, c\}$, B $= \{d, e, f, g\}$ and $f = \{(a, d), (b, e), (c, f)\}$

(*ii*) Where, X $= \{1, 2, 3\}$, Y $= \{1, 4, 9, 10, 25\}$, and

$f = \{(1, 1), (2, 4), (3, 9)\}$

6. One-one onto function or Bijective function

A function that satisfies the characteristics of both the 1-1 and onto function as under is called one-one onto function or a *Bijective function :*

(*i*) each element of the domain is associated with a separate element of the domain.

(*ii*) no element of the co-domain remains unrelated with an element in the domain.

Examples :

(i) Where A $= \{c, h, a, r, m\}$, B $= \{m, a, r, c, h\}$ and

f : A $\to$ B $= \{(c, m), (h, a), (a, r), (r, c), (m, h)\}$

(ii) Where, A $= \{1, 3, 5, 7\}$, B $= \{1, 27, 125, 343\}$ and

f : A $\to$ B $= \{(1, 1), (3, 27), (5, 125), (7, 343)\}$

7. Many-one into function : $f(x)$

A function that satisfies the characteristics of both many-one and into functions as under is called many-one into function :

(i) two or more elements of the domain are related to one element of the co-domain,

(ii) there remains atleast one element in the co-domain which is not associated with any element of the domain.

Examples

(*i*) Where, P = {h, a, r, m}, Q = {f, a, r, m, e, r} and

f: P → Q = (h, a), (a, a), (r, r), (m, m)}

(*ii*) Where, S = {1, 2, 3, 4}, T = {1, 4, 9, 10, 12} and

f : S → T = {(1, 1), (2, 4), (3, 9), (4, 9)}

8. Many-one onto function

A function is said to be a many-one onto functions, when it satisfies the characteristics of both the many-one and onto functions as under :

(*i*) two or more elements of the domain are related to one element of the co-domain.

(*ii*) no element of the co-domain is left unrelated with any element of the domain.

Examples :

(*i*) Where, A = {1, 2, 3, 4}, B = {3, 6, 9}, and

f : A → B = {(1, 3), (2, 3), (3, 6), (4, 9)}

(*ii*) Where, C = {*c, r, e, d, i, t*}, D = {*c, a, s, h*} and

f : C → D = {(*c, c*), (*r, a*), (*e, s*), (*d, h*), (*i, h*), (*t, h*)}

9. Constant function

A function is said to be a constant one when all the elements of its domain are associated with a single element of its co-domain.

Example :

(*i*) Where, A = {1, 2, 3}, B = {4, 5, 6, 7, 8}, and *f* = {(1, 6), (2, 6), (3, 6)}

10. Inverse function

A function that is obtained by interchanging the ordered pairs of a one-one onto function is called an inverse function. It is denoted by f^{-1} : B → A, or by *g*(*y*). Thus, where, A = {1, 2, 3}, B = {*a, b, c*} and *f* : A → B = {(1, *a*), (2, *b*), (3, *c*)}, the inverse function or f^{-1} : B → A = {(*a*, 1), (*b*, 2), (*c*, 3)}.

It must be noted that to obtain an inverse function from a given function the following conditions must be fulfilled :

(i) The number of elements in the domain of the function must be equal to those of its co-domain.

(ii) The given function must be of the type of 1 – 1 and onto.

If any of the above two conditions is not satisfied, then there can not exist an inverse of the given function.

The examples of such a function can be taken as under :

	Function (f)	*Inverse function* f^{-1}
(*i*)	If $y = x^2$	Then $x = \pm\sqrt{y}$
(*ii*)	If $y = x - 5$,	Then $x = y + 5$
(*iii*)	If $y = \frac{5x+3}{2x+9}$,	Then $x = \frac{3-9y}{2y-5}$
(*iv*)	If $y = \frac{ax+b}{cx+d}$,	Then $x = \frac{dy-b}{a-cy}$
(*v*)	If $x = \sqrt[3]{y}$,	Then $y = x^3$.

II. On the basis of theory of number system

(a) Real value function

When a function is a set of some real numbers, it is called a real value function. A real value function is of three types, *viz.* : (*i*) Single value function, (*ii*) Double value function, and (*iii*) Multivalue function.

When a function has only one value corresponding to each value of the independent variable (*i.e.*, of domain), it is called a single or one value function, *e.g.*, $y = x^2$. When a function has two values corresponding to each value of the independent variable, it is called a two or double value function, *e.g.*, $y = \sqrt{x}$, *i.e.*, $+\sqrt{x}$ and $-\sqrt{x}$. When a function has several values corresponding to each value of the independent variable, it is called a Multi value or Many valued function, *e.g.*, $y = x^2 + 3x - 7$.

(b) Absolute value function

A function is said to be an absolute value function or a **Modulus function** when it is defined as follows :

$$y = f(x) = \begin{cases} x, \text{for } x > 0 \\ 0, \text{for } x = 0 \\ -x, \text{for } x < 0 \end{cases}$$

(c) Explicit function

A function that is expressed directly in terms of the independent variable is called an Explicit function. For example, $y = x^2 + 7x - 4$.

(d) Implicit function

A function that is not expressed directly in terms of the independent variable and in which either of the two variables determines the other, is called an implicit function. For example, $4x - 5y = 0$, where $x = \frac{5y}{4}$, and $y = \frac{4x}{5}$.

(e) Continuous function

A function is said to be continuous at a point $x = a$, if $f(x)$ possesses a finite and definite limit as x tends to the value, 'a' from either lower or upper side of 'a', say 3 *viz.* : 2·9, 2·99 etc. or 3·1, 3·01 etc. respectively, and each of these limits is equal to $f(a)$.

(f) Discontinuous function

If any of the essential conditions of a continuous function is not satisfied, then the function is called a Discontinuous function.

(g) Even function

A function that satisfies $f(-x) = f(x)$ is called an Even function. The examples of such a function are : $f(x) = 5x^2 + \cos x$; $f(x) = 7x^4 - 5x^2 + 3$; $f(x) = \cos x$.

(h) Odd function

A function that satisfies $f(x) = -f(x)$ is called an odd function. The examples of such a function are : $f(x) = \sin x$; $f(x) = x^3$; $f(x) = 5x + 7x^3$ etc.

(i) Periodic function

A function in which the range can be separated into equal sub-intervals in such a manner that the graph of the function remains the same in each of the part intervals, is called a periodic function.

(j) Composite function

A function is said to be a composite function or a function of functions, when it becomes a function of another function. Thus, if $y = g(u)$, $u = f(x)$, and y or $gof = g\{f(x)\}$, then y is a function of function or composite function. For example, the volume of a tank is the function of its area, while the area itself is the function of its radius (*i.e.* $A = \pi . r^2$), and so the volume is the function of the radius of the area.

The phenomenon of a composite function can be better understood from the following arrow diagram

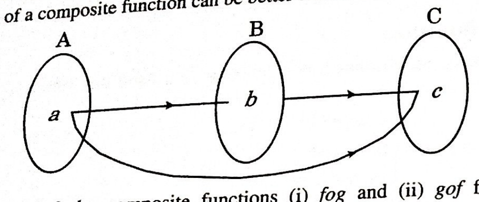


ILLUSTRATION 1. Find the composite functions (i) *fog* and (ii) *gof* from the following two functions :

(a) $f(x) = x^2 + 3, x \in R$ and

(b) $g(x) = x + 3, x \in R$

Also, comment on the result.

SOLUTION

(i) $fog = f\{g(x)\} = f(x + 3)$

$= (x + 3)^2 + 3 = x^2 + 6x + 9 + 3 = x^2 + 6x + 12$

(ii) $gof = g\{f(x)\} = g(x^2 + 3) = x^2 + 3 + 3 = x^2 + 6$

Comment

From the above results it is observed that $f\{g(x)\} \neq g\{f(x)\}$.

(k) Algebraic function

An algebraic function is one, the variables x, y etc. of which are finite in number and are affected by the operations of addition, subtraction, multiplication, division, power, and roots.

The examples of such functions are :

(i) $y = 3x^2 + 2x - 7$,

(ii) $y = \frac{1}{x^3} - \sqrt{x}$, etc.

(l) Transcendental function

A function that is not algebraic in nature is called a transcendental or non-algebraic function. Such functions include exponential functions, logarithmic functions, trigonometric functions, inverse trigonometric functions etc. which are explained in the paragraphs to follows :

(m) Trigonometric function

A non-algebraic function in which the trigonometric ratios (*i.e.,* the relations between any two of the three sides of a triangle) are used is called a Trigonometric function.

The examples of such a function are :

$y = \sin x$, $f(x) = \cos x$, $y = \text{cosec}^2 (x + 2)$, $y = \tan x$, and the inverse trigonometric functions like, $y = \sin^{-1} x$, $f(x) = \cos^{-1} x$ etc.

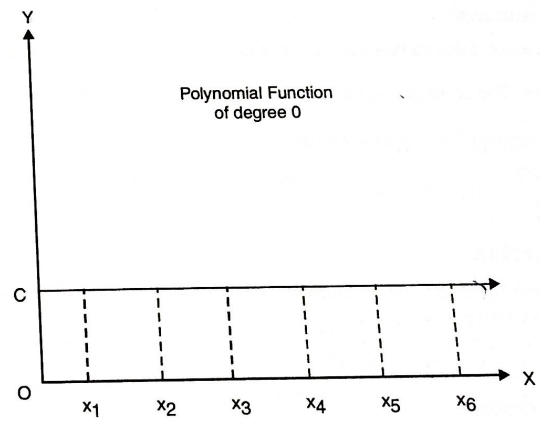
(n) Polynomial function

A function of the form $f(x) = ax^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ is called a polynomial function, where 'n' is a non-negative integer and $a_0, a_1 \ldots a_n$ are constants with $a_0 \neq 0$. Here a_0 is called the leading coefficient. For example, $f(x) = 2x^3 - 5x^2 + 7x + 1$ is a polynomial function in x of degree 3. It may be noted that a linear function is a polynomial function of degree 1 and Quadratic function is a polynomial of degree 2. The doman of any polynomial is a set of all real numbers. Such polynomial functions are of various types such as (*i*) Constant function of degree 0, (*ii*) Linear function or polynomial function of degree 1, (*iii*) Quadratic function or polynomial functions of degree 2 and (*iv*) Cubic function or polynomial function of degree 3. These are briefly identified as under :

(i) Constant function or Polynomial function of degree, 0

A function of x of the form $f(x) = ax^0$ or $f(x) = 3$ is called a constant polynomial function of degree, 0. Here, the constants a, c etc. are real numbers and $x^0 = 1$. In such a function, the range consists of only one element.

Graphically such a function would be represented as under :

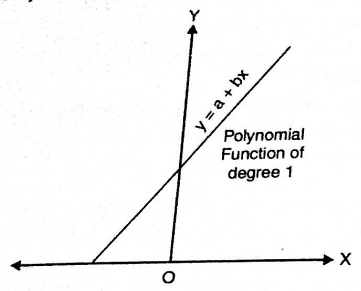


(Constant function)

(ii) Linear function or Polynomial function of degree, 1

A function of the form, $f(x) = 3x - 5$, $y = a + bx$, $y = 5x + 7$ etc. is called a linear function of degree, 1, where $x^1 = x$ and the constants *viz.* : a, c, etc. are real numbers.

When presented graphically, such a function takes the following shape :



(Linear function)

(iii) Quadratic function of degree, 2

A polynomial function of the form, $y = 3x^2 - 2x + 5$, in which the highest power of the independent variable, x is 2 is called a Quadratic function. In such a function, the constants like, a, b, c, etc. are all real numbers and not equal to zero.

(iv) Cubic function of degree, 3

A polynomial function of the form $y = x^3 + 2x^2 - 3x + 7$, in which the highest power of the independent variable x is 3 is called a Cubic function of degree 3. In such a function the constants like, a, b, c etc. are all real numbers and not equal to zero.

(o) Rational function

An algebraic function that can be expressed as a ratio of two polynomials is called a Rational function. The form of such a function is $r(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The examples of such a function are :

(i) $r(x) = \dfrac{x^2 5x+2}{x^2+3}$, (ii) $f = \dfrac{x^2+5}{3x^2+7}$ etc.

(p) Irrational function

A function in which the independent variable, x has root extraction of terms is called an irrational function. The examples of such a function are :

(i) $f(x) = \sqrt{x}$, (ii) $y = \sqrt{x^2 + 5x + x3 + 7x5}$

(q) Monotonic function

A function in which the dependent variable changes with a change in the value of the independent variable is called a Monotonic function. When the dependent variable increases with an increase in the independent variable the function is called a monotonically increasing function. For example, the supply as a function of price. But when the dependent variable decreases with an increase in the independent variable, it is called a monotonically decreasing function. For example, the demand as a function of Price.

Thus, the function, $f : x \to 2x$, $x \in R$ is a monotonic increasing function, and $f : x \to \dfrac{1}{x}$ is a monotonic decreasing function, where $x > 0$.

(r) Identity function

A function in which each element of the domain corresponds to itself is called an Identity function. The expression of such a function runs as under :

$$f(x) = \{(1, 1), (2, 2), (3, 3)\}$$

(s) Signum function

A function is called a signum function when it is defined as follows :

$$\text{Sgn. } x = \begin{cases} +1, \text{ when } x>0 \\ 0, \text{ when } x=0 \\ -1, \text{ when } x<0 \end{cases}$$

Such a function is denoted by Sgn $x = \dfrac{|x|}{x}$, where $x \neq 0$.

(t) Greatest Integer function

A function is called the Greatest integer function when it is expressed as $f(x) = [x]$ for the greatest integer less than or equal to x.

The examples, of such a function are :

$$f(2\cdot9) = 2, f(0\cdot7) = 0, f(-2\cdot9) = -3.$$

The elements of the function with the above element of the domain will be as under :

$$f = \{(2\cdot9, 2), (0\cdot7, 0), (-2\cdot9, -3)\}.$$

Such functions are also, otherwise called *Step functions*.

(u) Equal function

A function is said to be an equal to another function, if and only if its domain and range are equal to the corresponding domain and range of that another function.

Thus, $f(x) = g(x)$, where, $d(f) = d(g)$ and $r(f) = r(g)$

(v) Parametric function

A function is said to be a parametric one when both of its independent variable, x and dependent variable, y are expressed in terms of another third variable.

Thus, if $y = f(t)$, where, $x = f(t)$ and $y = t^2 + 1$ where $x = 2t$ are the examples of parametric function.

3. OTHER MATHEMATICAL FUNCTIONS

(a) Linear Functions

(b) Quadratic Functions

(c) Exponential Functions

(d) Logarithmic Functions

(e) Logistic Functions

(a) Linear Function

A function of the form $f(x) = ax + b$ where a and b are constants is called a linear function. The graph of the function $y = ax + b$ is a straight line in the XY plane. Here, 'a' is called the slope of the straight line and b is called the intercept on the y-axis. If $b = 0$, the graph is a straight line passing through the origin. If $a = 0$, the graph is a straight line parallel to the X-axis and a distance b units from the X-axis.

A straight line can be expressed as follows :

(i) $y = mx + c$, called slope intercept form.

(ii) $\frac{x}{c} + \frac{y}{b} = 1$ (here 'a' and 'b' intercepts made on X-axis and Y-axis.)

(iii) $(y - y_1) = m(x - x_1)$. This is the straight line passing through the point (x_1, y_1) and having slope 'm'.

(iv) $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$. This is the equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2).

ILLUSTRATION 2. The salary of an employee in 2015 was ₹ 1,20,000. In 2017, it will be ₹ 1,35,000. Express the salary as a linear function of time and estimate his salary in 2018.

SOLUTION

Let S represent the salary (in ₹) and t represent the year. Thus,

Year		Salary	
2015	t_1	1,20,000	S_1
2017	t_2	1,35,000	S_2
2018	t	–	S

The equation of straight line represents salary as a linear function of time and,

$$\frac{S - S_1}{t - t_1} = \frac{S_1 - S_2}{t_1 - t_2} \text{ or } \frac{S - 1,20,000}{2018 - 2015} = \frac{1,20,000 - 1,35,000}{2015 - 2017}$$

$$\Rightarrow \quad S - 1,20,000 = \frac{-15,000}{-2} \times 3$$

$$\Rightarrow \quad S = 7,500 \times 3 + 1,20,000 = 22,500 + 1,20,000 = ₹\ 1,42,500$$

∴ The estimated salary in 2018 is ₹ 1,42,500.

ILLUSTRATION 3. The life expectancy of males in 2011 in a country is 70 years. In 2001, it was 60 years. Assuming the life expectancy to be a linear function of time, make a prediction of the life expectancy of males in that country in 2016.

SOLUTION

Let E be the life expectancy and t the year

Year		Expectancy	
2001	t_1	60	E_1
2011	t_2	70	E_2
2016	t	–	E

The linear function representing the life expectancy in the year t is

$$\Rightarrow \quad \frac{E - E_1}{t - t_1} = \frac{E_1 - E_2}{t_1 - t_2} \quad \text{or} \quad \frac{E - 60}{2016 - 2001} = \frac{60 - 70}{2001 - 2011}$$

$$\Rightarrow \quad \frac{E - 60}{15} = \frac{-10}{-10} \quad \text{or} \quad E - 60 = 1 \times 15$$

$$\Rightarrow \quad E = 1 \times 15 + 60 = 75$$

The Expected life expectancy in the year 2016 is 75 years.

ILLUSTRATION 4. The demand curve for a commodity is $x = 100 - \frac{y}{4}$, where y is the price per unit and x is the number of units demanded.

(i) Find the quantity demanded if price is ₹ 40.

(ii) Find the price if the quantity demanded is ₹ 7.

(iii) What quantity would be demanded if the commodity were free ?

SOLUTION

(i) The linear relationship between price and quantity is

$$x = 100 - \frac{y}{4}$$

When $y = 40$, $x = 100 - \frac{40}{4} = 100 - 10 = 90$

Thus, when price is ₹ 40, the quantity demanded is 90 units.

(ii) When $x = 70$, $70 = 100 - \frac{y}{4}$ or $y = 30 \times 4 = 120$

Thus, when the quantity demanded is ₹ 70, the price per unit is ₹ 120.

(iii) when $y = 0$, $x = 100 - \frac{0}{4} = 100$

Thus, when the commodity were free, the quantity demanded will be 100 units.

(b) Quadratic Function

A function of the form $f(x) = ax^2 + bx + c$ where, a, b and c are constants is called a quadratic function or a polynomial of degree 2.

A quadratic polynomial can be expressed as the product of two linear factors. Some of the properties of quadratic functions are stated as below :

$$f(x) = ax^2 + bx + c$$

$$= a\left[x^2 + \frac{bx}{a} + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2\right]+a\left[\frac{4ac-b^2}{4a^2}\right]$$

$$= a\left(x+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a}\right) \qquad \ldots(I)$$

If a is a positive number, $f(x)$ takes its minimum value at $x=-\frac{b}{2a}$.

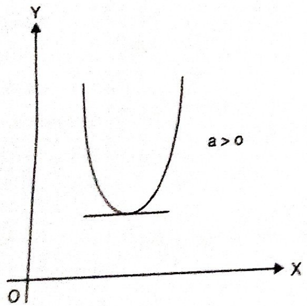
If a is negative, $f(x)$ takes its maximum value at $x=-\frac{b}{2a}$.

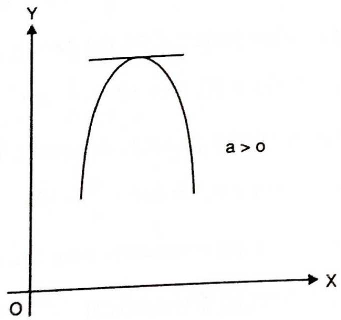
The graph of the function $f(x) = ax^2+bx+c$ is called a parabola. The point at which minimum or maximum value occurs is called the vertex of parabola.

The vertex is $\left(\frac{-b}{2a}, \frac{-D}{2a}\right)$

Where, $\quad D = b^2-2ac$.

The graph of $f(x)$ when $a>0$ and when $a<0$ is depicted as below :





Thus, given a second degree polynomial

$$ax^2+bx+c.$$

To find the maximum or minimum point on the graph of the function, the following steps are to be taken up.

Step–I Find the values of a, b and c.

Step–II Determine whether $a>0$ or $a<0$.

Step–III If $a>0$, the minimum point occurs at $x=\frac{-b}{2a}$. Substituting this value of x in $f(x)$, obtain the minimum value of $f(x)$. If $a<0$, the maximum point occurs at $x=\frac{-b}{2a}$. Substituting this value of x in $f(x)$ we obtain the maximum value of $f(x)$.

Roots of Quadratic Equation

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-2ac}{4a}\right)$$

The roots of $ax^2+bx+c=0$, are given by $a\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-2ac}{4a}\right)=0$.

$$\Rightarrow \quad a\left(x+\frac{b}{2a}\right)^2 = \left(\frac{b^2-2ac}{4a}\right)$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-2ac}{4a^2}$$

$$\Rightarrow \quad x+\frac{b}{2a} = \sqrt{\frac{b^2-2ac}{4a^2}}$$

Thus,

$$x = -\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Here, the roots are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$

Remember :

(i) If $b^2-4ac>0$, the roots are real.

(ii) If $b^2-4ac=0$, the roots are real and equal.

(iii) If $b^2-4ac<0$, the roots are imaginary.

ILLUSTRATION 5. The total profit in rupees of a drug company from the manufacture and sale of x drug bottles is given by

$$y = \frac{-x^2}{400}+2x-80$$

(a) How many drug bottles must the company sell to achieve the maximum profit ?

(b) What is the profit per drug bottle when this maximum is achieved ?

SOLUTION

Given, $y = \frac{-x^2}{400}+2x-80$

Comparing with the quadratic function, we get

$$y = ax^2+bx+c$$

We have, $\quad a = -\frac{1}{400}, b=2$ and $c-80$

Here, a is negative. Thus, y is maximum when.

$$x = \frac{-b}{2a} = \frac{-2}{2\times\frac{-1}{400}} = 400\times\frac{-2}{-2} = 400$$

$\therefore$ To get the maximum profit, the value of $x = 400$ is substituted in the equation

$$\text{Maximum Profit} = \frac{-x^2}{400} + 2x - 80$$

$$= \frac{-(400)^2}{400} + 2\times400 - 80$$

$$= -400 + 800 - 80 = 320$$

$$\text{Maximum Profit per bottle} = \frac{320}{400} = ₹\,0.80$$

ILLUSTRATION 6. The price 'p' per unit at which a company can sell all that it produces is given by the function

$$p(x) = 300 - 4x.$$

The cost function is $C(x) = 500 + 28x$, where x is the number of units produced. Find 'x' so that the profit is maximum.

SOLUTION

Given,

$$C(x) = 500 + 28x$$
$$p(x) = 300 - 4x$$
$$\text{Revenue } (x) = x.\,p(x) = x\,(300 - 4x) = 300x - 4x^2$$
$$\text{The profit function } P(x) = R(x) - C(x)$$
$$= (300x - 4x^2) - (500 + 28x)$$

$$\therefore \quad P(x) = -4x^2 + 272x - 500$$

Comparing the quadratic function $ax^2 + bx + c$

Here, $a = -4, b = 272$ and $c = -500$

Since, $a < 0$, the profit is maximum when $x = \frac{-b}{2a}$.

$$\text{Thus, } x = \frac{-272}{2x-4} = \frac{-272}{-8} = 34$$

$\therefore$ For maximum profit, the company should produce and sell 34 units.

ILLUSTRATION 7. A manufacturer finds that the cost per unit of manufacturing a certain commodity is $y = x^2 - 20x + 200$, when 5 to 200 units are produced per day. Find the number of units to be produced when the cost is least and also find the cost per unit.

SOLUTION

The cost function is given by

$$C(x) = x^2 - 20x + 200$$

This is a quadratic function of the form $ax^2 + bx + c$

Here, $a = 1$, $b = -20$ and $c = 200$

Since, $a > 0$, the cost per unit is minimum when

$$x = \frac{-b}{2a} = \frac{-(-20)}{2\times1} = \frac{+20}{2} = 10$$

For the minimum cost, the manufacturer should produce 10 units.

Cost per unit can be find out by substituting the value of x in the cost function

$$C(x) = x^2 - 20x + 200$$
$$= (10)^2 - 20\times10 - 200 = 100$$

The minimum cost per unit is ₹ 100.

(c) Exponential Function

A function having a variable base and a constant exponent is called a power function i.e. $y = x^3$ is a power function.

Any polynomial in x is a power function. A function having a constant base and a variable exponent is an **exponential function.** Some of the properties of the exponential functions are stated below. These properties are also true for power functions.

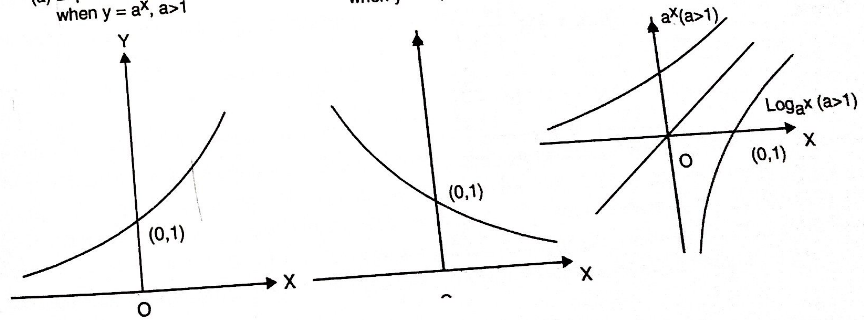
1. $a^x.a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x.b^x$
5. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} (b \neq 0)$
6. $a^{-x} = \frac{1}{a^x}$
7. $\sqrt[y]{a^x} = (a)^{x/y}$ where x and y are integers, $x > 0$.

The simplest exponential function is $y = a^x$, where $a > 0$, $a \neq 1$ and the exponent x is any real number. The function $f(x) = a^x$ is called an exponential function to the base 'a'. For example, $y = 2^x, y = 3^x$ and $y = \left(\frac{1}{2}\right)^x$ are exponential functions.

(a) Exponential function when $y = a^x$, $a>1$

(b) Exponential function when $y = a^x$, $a<1$

(c) Logarithmic Function when $y = a^x$, $a>1$ and $y = \log_a X$, $a>1$



The most used exponential function is $y=e^x$, which 'e' is an irrational number whose value lies between 2 and ($e = 2.718$ approx.)

The precise definition of base e and its advantage can be find out in the logarithmic function.

(d) Logarithmic Function

The logarithm of a positive number y to a base 'a' ($a \neq 1$) is the exponent to which the base is raised to equal the number.

i.e., if $y=a^x$ then $\log_a y = x$

Here, if y is an exponential function, x is a logarithmic function.

The exponential function and the logarithmic function are inverse functions of each other. The logarithmic functions have the following properties

(i) $\log_a xy=\log_a x+\log_a y$

(ii) $\log_a \frac{x}{y}=\log_a x-\log_a y$

(iii) $\log_a x^n = n\log_a x$

(iv) $\log_a x=\log_b x \times \log_a b$

The graph of $y=\log_a x, a>0$, entirely lies in the first two quadrants.

ILLUSTRATION 8(a). If a, b and c are any 3 consequentive integers, prove that

$$\log(1+ac) = 2\log b$$

SOLUTION

Since, a, b, c are 3 consequentire integers they are $a = b - 1$ and $c = b + 1$

$$ac = (b-1)(b+1) = b^2 - 1$$

$$1 + ac = 1 + b^2 - 1 \text{ or } 1 + ac = b^2$$

$$\Rightarrow \quad \log(1+ac) = 2\log b \quad \text{(Proved)}$$

ILLUSTRATION 8(b). If $f(x)=\log\frac{1+x}{1-x}$, show that $2f(x)=f\left(\frac{2x}{1+x^2}\right)$.

SOLUTION

Given, $$f(x) = \log\left(\frac{1+x}{1-x}\right)$$

$$f\left(\frac{2x}{1+x^2}\right) = \log\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}$$

$$= \log\frac{1+x^2+2x}{1+x^2-2x}$$

$$= \log\frac{(1+x)^2}{(1-x)^2}=\log\left(\frac{1+x}{1-x}\right)^2$$

$$= 2\log\left(\frac{1+x}{1-x}\right)=2f(x) \quad \text{(Proved)}$$

ILLUSTRATION 9. If $x^2+y^2=6xy$ show that, $2\log(x+y)=\log x+\log y+3\log 2$

SOLUTION

Given, $$x^2+y^2 = 6xy$$

LHS, $$2\log(x+y) = \log(x+y)^2=\log(x^2+y^2+2xy)$$

$$= \log(6xy+2xy)=\log 8xy$$

$$= \log 8+\log x+\log y$$

$$= 3\log 2+\log x+\log y = \log x + \log y + 3\log 2 \quad \text{(Proved)}$$

ILLUSTRATION 10. A sum of rupees 5,000 is invested in a bank that pays 10% interest. The interest is compounded annually. Find the amount at the end of 5 years.

SOLUTION

Interest is compounded annually.

$$A = P(1+i)^n$$

$$P = ₹\,5{,}000;\ i = \frac{10}{100} = 0.1 \text{ and } n = 5$$

$$A = 5{,}000(1+0.1)^5$$

$$= 5{,}000(1.1)^5$$

$$\log A = \log 5{,}000 + 5\log 1.1$$

$$= 3.6990 + 5(0.0414)$$

$$= 3.906$$

$$\therefore \quad A = AL\ 3.906 = ₹\,8{,}054$$

Thus, the amount at the end of 5 years is ₹ 8,054.

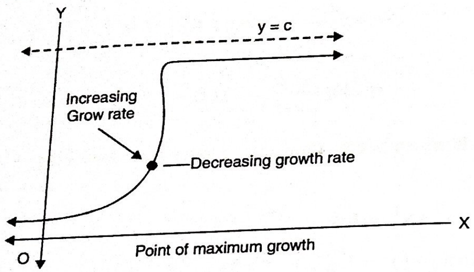
(e) LOGISTIC FUNCTION

The general equation of a logistic function is given by

$$f(x) = \frac{c}{1+ae^{-rx}}$$

where, a, c and r are all positive constants.

The graph of logistic function look like a stretched S-shape and are increasing or decreasing between two horizontal line. There is one change in curvature. Data that follows an increasing logistic curve usually describes constrainted growth or cumulative quantity. For small values of the independent variable the increasing logistic function behaves very much likely an (increasing) exponential function. However, for large values of the independent variable, the two functions behave quite differently.



Logistic growth functions are used in the model of real life quantities, whose growth levels off because the rate of growth changes–from an increasing growth rate to a decreasing growth rate.

ILLUSTRATION 11. Evaluate : $f(x)=\dfrac{100}{1+9e^{-2x}}$ for each values of x

(a) $f(-3)$, (b) $f(0)$, (c) $f(2)$, (d) $f(4)$

SOLUTION

(a) $f(-3)=\dfrac{100}{1+9e^{-2\times -3}}=\dfrac{100}{1+9e^{6}}=\dfrac{100}{1+9\times 403.03}=\dfrac{100}{3631.87}=0.275$

(b) $f(0)=\dfrac{100}{1+9e^{-2\times 0}}=\dfrac{100}{1+9}=10$

(c) $f(2)=\dfrac{100}{1+9e^{-2\times 2}}=\dfrac{100}{1+9e^{-4}}=\dfrac{100}{1+9\times 0.01832}=\dfrac{100}{1.1648}=85.8$

(d) $f(4)=\dfrac{100}{1+9e^{-2\times 4}}=\dfrac{100}{1+9e^{-8}}=\dfrac{100}{1+9\times 0.00034}=\dfrac{100}{1.00306}=99.7$

Note. The value of e^{x} and e^{-x} can be traced from the table value appended in the book.

Characteristics of Logistic Function

1. The y-intercept is $\dfrac{c}{1+ae^{-rx}}$
2. The horizontal lines $y=0$ and $y=c$ are equidistance.
3. The domain is all real numbers and the range is $0<y<c$.
4. The graph is increasing from left to right.

ILLUSTRATION 12. Solve : $\dfrac{50}{1+10e^{-3x}}=40.$

SOLUTION

$$\frac{50}{1+10e^{-3x}}=40$$

$$\Rightarrow \quad 50 = 40\,(1+10e^{-3x})$$
$$\Rightarrow \quad 50 = 40+400\,e^{-3x}$$
$$\Rightarrow \quad 10 = 400\,e^{-3x}$$
$$\Rightarrow \quad e^{-3x} = \frac{10}{400}=0.025$$
$$\Rightarrow \quad -3x\log e = \log 0.025$$
$$\Rightarrow \quad -3x = \log 0.025$$
$$\Rightarrow \quad x = \frac{-1}{3}\log 0.025 = -\frac{1}{3}\times\bar{2}.3979 = -\frac{1}{3}\times -1.6021$$
$$= 0.5340$$

(**Note.** The value of $\log_e e = 1$)

EXERCISE (A)

1. A firm produces 200 units of a product for a total cost of ₹ 730 and 500 units of product for a total cost of ₹ 970. Assuming the cost curve to be linear, derive the equation and derive the equation of this straight line and also use it to estimate the cost of producing 400 units of the product. (**Ans.** $y=0.8x+570$, ₹ 890)
2. A firm produces 50 units of product for ₹ 320 and 80 units for ₹ 380. Assuming that the cost function is linear, estimate the cost of producing 10 units. (**Ans.** 240)
3. The supply curve of a commodity is $x = 11y - 1$ where y represent price and x represents quantity supplied. Find
   (i) the price when the quantity supplied as 54, and
   (ii) the quantity when the price per unit is ₹ 2. (**Ans.** ₹ 5, ₹ 21)
4. Cost of manufacturing a particular item is given by $C=x^2-70x+2500$, where x denotes the number of units of the item and C is the cost per item. How many units the company should produce in order that the cost per unit is the minimum. What is the minimum cost per unit ? (**Ans.** 35, ₹ 1,275)
5. Determine the profit or minimum value of the following functions :
   (i) $-2x^2+8x+3$
   (ii) $3x^2-12x+5$ (**Ans.** 11, –7)
6. Determine the value of the output at which the cost function, $y=x^2-6x+120$ is minimum. (**Ans.** 3)
7. The total profit y in rupees that a company can make by selling x units of a commodity is given by $y=\dfrac{-x^2}{100}+20x-5{,}000$.
   (i) How many units the company should sell to get the maximum profit ?
   (ii) What is the maximum profit per unit of the commodity ? (**Ans.** 1000, ₹ 5)

8. The total revenue function for a commodity x is given by $y = 12x + \frac{x^2}{2} - \frac{x^3}{3}$ Determine:

(i) Output at which marginal revenue is maximum.

(ii) The output at which the average revenue is maximums.

(iii) The output at which the average revenue is equal to marginal revenue. (**Ans.** $\frac{1}{2}, \frac{3}{4}, \frac{3}{8}$)

9. Satish invests ₹ 35,000 in a bank which pays 8% interest compounded yearly. What is the amount due after the end of 3 years ? (**Ans.** ₹ 44065)

10. A sum invested at 8% in a bank becomes treble in a certain number of years. Determine the number of years. (**Ans.** 14 yrs.)

11. Evaluate the function $f(x) = \frac{12}{1+5e^{-2x}}$ for the given value of x

(a) $f(0)$ (b) $f(-2)$ (c) $f(5)$ (d) $f\left(-\frac{1}{2}\right)$ (**Ans.** (a) 12 (b) 0.0437 (c) 11.99 (d) 0.82)

12. Solve the equation

(a) $\frac{18}{1+2e^{-2x}} = 10$, (b) $\frac{30}{1+4e^{-x}} = 10$, (c) $\frac{12.5}{1+7e^{-0.2x}} = 9$

4. ALGEBRA OF FUNCTIONS

By algebra of functions we mean the application of the four basic operations of addition, subtraction, multiplication, and division on any two or more functions. It is performed just in the same manner as it is done in case of simple arithmetic.

Thus, if $f(x)$ and $g(x)$ are any two functions, then their four algebraic operations will be performed as under :

(*i*) Addition of $f(x)$ and $g(x) = f(x) + g(x)$

(*ii*) Subtraction of $g(x)$ from $f(x) = f(x) - g(x)$

(*iii*) Multiplication of $f(x)$ and $g(x) = f(x) \cdot g(x)$

(*iv*) Division of $f(x)$ by $g(x) = \frac{f(x)}{g(x)}$

The following illustrations will show how algebra of functions are performed.

ILLUSTRATION 13. Perform the four basic operations on the following two functions :

(i) $f(x) = 2x^2 - 5$, (ii) $g(x) = 3x + 5$

SOLUTION

(*i*) Addition : $f(x) + g(x) = (2x^2 - 5) + (3x + 5)$

$= 2x^2 - 5 + 3x + 5 = 2x^2 + 3x$

(*ii*) Subtraction : $f(x) - g(x) = (2x^2 - 5) - (3x + 5) = 2x^2 - 5 - 3x - 5 = 2x^2 - 3x - 10$

(*iii*) Multiplication : $f(x).g(x) = (2x^2 - 5)(3x + 5)$

$= 6x^3 - 15x + 10x^2 - 25 = 6x^3 + 10x^2 - 15x - 25$

(*iv*) Division : $\frac{f(x)}{g(x)} = \frac{(2x^2-5)}{(3x+5)} = \frac{2x^2-5}{3x+5}$

ILLUSTRATION 14. If $f(x) = [x]$, $x \in R$, and $g(x) = x$, where $x \in R$, find

(i) $f + g$, (ii) $f - g$, (iii) $f.g$, and (iv) $\frac{f}{g}$.

SOLUTION

Given, $f(x) = [x]$, and $g(x) = x$

Thus,

(*i*) $f(x) + g(x) = x + x = 2x$, when $x > 0$

$= -x + x = 0$, when $x < 0 = 0 + x = x$, when $x = 0$.

(*ii*) $f(x) - g(x) = x - x = 0$, when $x > 0$

$= -x - x = -2x$, when $x < 0 = 0 - x = -x$, when $x = 0$

(*iii*) $f(x).g(x) = x.x = x^2$, when $x > 0$

$= -x.x = -x^2$, when $x < 0 = 0 \times x = 0$, when $x = 0$

(*iv*) $\frac{f(x)}{g(x)} = \frac{x}{x} = 1$, when $x > 0$

$= \frac{x}{x} = -1$, when $x < 0 = \frac{0}{x} = 0$, when $x = 0$.

ILLUSTRATION 15. The total cost function C(x) of producing x items is given by

(i) $C(x) = 1{,}000 + 5x$ when $0 \le x \le 500$

$C(x) = 2{,}000 + 4x$, when $500 < x \le 2{,}000$

Find the cost of producing (i) 430 items (ii) 1,200 items.

SOLUTION

(i) Given $C(x) = 1{,}000 + 5x \quad 0 \le x \le 500$

and $C(x) = 2{,}000 + 4x$, when $500 < x \le 2000$

for $x = 430$, $C(x) = 1{,}000 + 5x$

$C(430) = 1000 + 5 \times 430 = 1{,}000 + 2{,}150 = 3{,}150$

Again for $x = 1{,}200$, $C(x) = 2{,}000 + 4x$

$C(1{,}200) = 2{,}000 + 4 \times 1{,}200 = 6{,}800$

EXERCISE (B)

1. Write short notes on the following :

(*i*) Even function, (*ii*) Odd function, (*iii*) Periodic function, (*iv*) Composite function, (*v*) Algebraic function, (*vi*) Transcendental function, and (*vii*) Parametric function.

2. Explain the following functions along with their notations :

(*i*) Exponential function, (*ii*) Logarithmic function, (*iii*) Trigonometric function, (*iv*) Polynomial function, (*v*) Linear function, (*vi*) Quadratic function, (*vii*) Cubic function and (*viii*) Logistic function.

3. From the following point out which are the functions from A to A and which are not :

(*i*) $y = x$, (*ii*) $y = \sqrt{x}$, (*iii*) $y = \{(1, 1), (2, 2), (7, 7)\}$,

(iv) $|x| + y = 0$, (v) $y = \sqrt{5+2}\,x^2$, (vi) $y = \dfrac{3x}{x3}$

Ans. [(i) (iii) and (iv) are functions, and (ii), (v) and (vi) are not]

4. Find fog for each of the following function :

(i) $f(x) = x^8$, $g(x) = 2x^2 + 1$ (ii) $f(x) = x$, $g(x) = \dfrac{1}{x}$, $x \neq 0$

(iii) $f(x) = x^2$, $g(x) = (x + 1)$

Ans. [(i) $(2x^2 + 1)^8$, (ii) $\dfrac{1}{x}$, $x \neq 0$, (iii) $(x + 1)^2$]

5. Find gof for each of the following :

(i) $f(x) = x^2 + 1$, $g(x) = \dfrac{1}{x+1}$ (ii) $f(x) = \sqrt{x}$, $g(x)\ 2x^2 + 1$.

Ans. [(i) $\dfrac{1}{x^2+2}$, (ii) $2x + 1$].

6. Point out which of the following functions are monotonic and which are not :

(i) $y = +\sqrt{x^2 - 2}$, $x \geq 3$.

(ii) $y = |x|$, (iii) $y = [x - 1]$, (iv) $y = x^3$, (v) $y = \dfrac{5}{4}x$.

Ans. [(i) and (v) are monotonic functions and the rest are not]

7. Given, A = {1, 2, 3, 4}, and B = {1 ...... 10}, find the (i) Domain, (ii) Codomain, (iii) Range, (iv) Image, and (v) Pre-image of the function, $f : A \to B$ given $f(x) = 2x - 1$.

Ans. [(i) A, (ii) B, (iii) {1, 3, 5, 7}, (iv) {1, 3, 5, 7} and (v) {1, 2, 3, 4}

8. Find the inverse of each of the following function if exists :

(i) $f(x) = x^4 + 1$, (ii) $f(x) = |x - 3| + 1$, (iii) $f(x) = +\sqrt{x}$, xy,

(iv) $f(x) = 5x^3$, (v) $f(x) = 2x + 3$, (vi) $f(x) - \dfrac{2}{3}\sqrt{9 - x^2}$ $-3 \leq x \leq 3$.

[**Ans.** (i), (ii), (vi) function does not exist (iii) $f^{-1}(x) = x^2$, $x > 0$, (iv) $f^{-1}(x) = \dfrac{1}{5^{1/3}}\,x^{1/3}$ (v) $x = \dfrac{y-3}{2}$]

9. From the following series of functions and inverse functions, arrange the pairs of functions and their respective inverse functions :

Functions	***Inverse functions***
(i) $x = y + 5$	(i) $y = x^2$
(ii) $x = \dfrac{dy - b}{a - cy}$	(ii) $y = \dfrac{ax + b}{cx + d}$
(iii) $x = \pm\sqrt{y}$	(iii) $y = x - 5$.

Ans. [(i) and (iv), (ii) and (iii), (iii) and (ii), (iv) and (i)]

5. GRAPHIC REPRESENTATION OF FUNCTIONS

When the ordered pairs of values of any two variables given by a function are represented through a graph drawn on a graph paper, it is called graphic representation of a function. The picture of a graph will

be different for the function set under different rules. However, for representing any function through a graph we are to go by the following steps in turn.

Steps

(i) Tabulate the ordered pairs of values of the two variables X and Y according to the rule or definition of the given function.

(ii) Draw the two lines of X abscissa, and Y ordinate on a graph paper marking the point of their intersection at 0.

(iii) Divide both the X line and the Y line into a number of equal parts on the basis of a natural scale so as to accommodate thereon the maximum values of the respective variables.

(iv) Plot the dots at the coordinated points of each of the pairs of the values of X and Y, and draw a curve by joining all these points in a freehand manner. The curve thus obtained is the required graph of the given function.

While drawing such a curve, the points for which the curve is not defined are to be excluded and encircled to indicate that they are not included in the graph. Such an occasion arises only when the function is a discontinuous one.

The following illustrations will show how various functions can be represented through graphs.

ILLUSTRATION 16. Make a graphic representation of the function given below :

$f(x) = 2 + 2x$.

SOLUTION

The given function $f(x) = 2 + 2x$ is a linear function as the index of the independent variable x is only 1. For representing such a function through a graph we need only two coordinating points to draw the curve of a straight line. Thus, we tabulate the two ordered pairs for the two variables x and y on the basis of $f(x) = 2 + 2x$ as under :

x	0	4
y	2	10

With the above ordered pairs, the graph of the function will be drawn as under :

Graphic representation of the linear function, $f(x) = 2 + 2x$

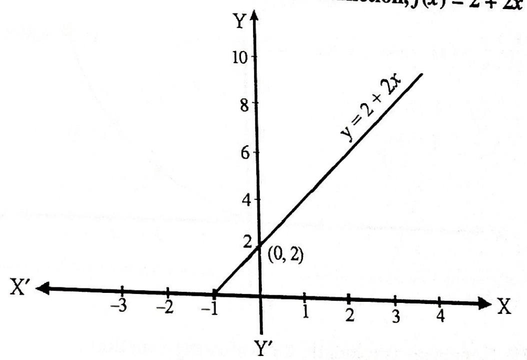


ILLUSTRATION 17. Represent graphically the function, $y = f(x) = 5$.

SOLUTION

The given function, $y = f(x) = 5$ is a constant function. In such a case, whatever may be the value of

the independent variable, x the value of the dependent variable y remains the same 5. Thus the required ordered pairs of values for such a function may be tabulated as follows :

x	0	1	2	3	4	5	6	...
y	5	5	5	5	5	5	5	5

With the above ordered pairs of values the required graph will be drawn as under :

Graphic representation of the Constant function $y = f(x) = 5$

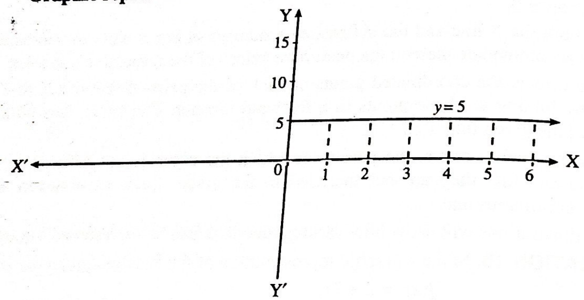


ILLUSTRATION 18. Draw the graph of the function $f(x) = 2^x$.

SOLUTION

The given function is an exponential one. The ordered pairs of values for such a function may be tabulated as under :

x	0	1	2	3	4
y	1	2	4	8	16

With the above pairs of values, the required graph will be drawn as under :

Graphic representation of the function $(fx) = 2^x$

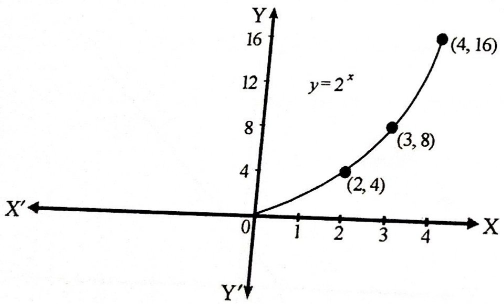


ILLUSTRATION 19. Represent graphically the following function :

$$y = f(x) = x^2$$

SOLUTION

The given function is a polynomial function of degree two. The ordered pairs of values for such a function may be tabulated as under :

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

With the above pairs of values of x and y, the required graph will be drawn as under :

Graphic representation of the function, $f(x) = x^2$

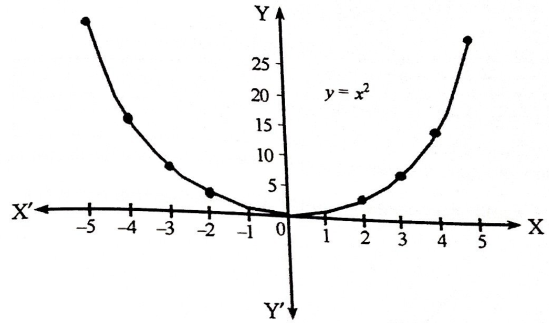


ILLUSTRATION 20. Represent graphically the function given below :

$$f(x) = \frac{x+1}{x-2}$$

SOLUTION

The given function is a rational one. The ordered pairs of values for such a function may be tabulated as follows :

x	–3	–2	–1	0	1	2	3
y	0·4	0·25	0	–0·5	–2	∞	4

With the above pairs of values the required graph will be drawn as under :

Graphic representation of the Rational function $(x) = \frac{x+1}{x-2}$

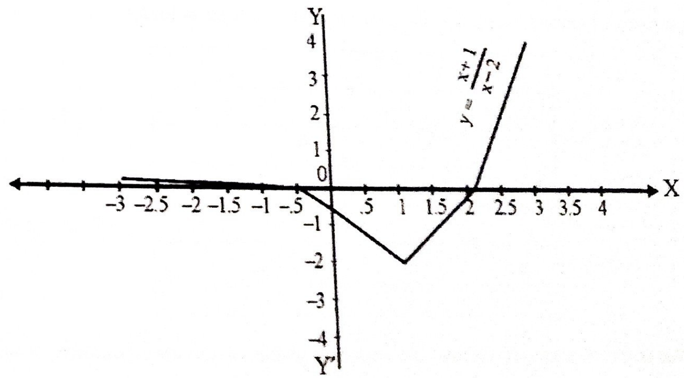


ILLUSTRATION 21. Draw the graph of the function $f(x) = |x|$.

SOLUTION

The given function is an absolute value or Modulus function which is expressed as under :

$$f(x) = \begin{cases} x \text{ for } x \geq 0 \\ -x \text{ for } x \geq 0 \end{cases}$$

The ordered pairs of values for such a function may be tabulated as under :

x	0	1	2	3	4	–1	–2	–3	–4
y	0	1	2	3	4	1	2	3	4

With the above pairs of values the required graph of the function may be drawn as follows :

Graphic representation of the Modulus function $f(x) = (x)$

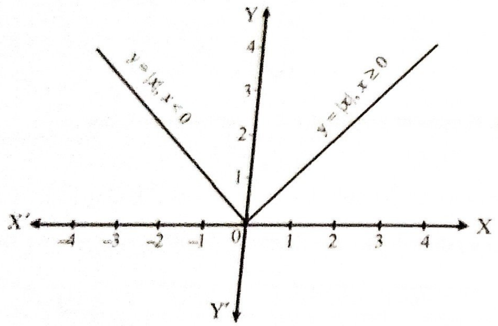


ILLUSTRATION 22. Represent the Greatest Integer function $y = [x]$ by means of a graph.

SOLUTION

The given function is a *step* function or a function of the greatest integer less than or equal to x. The required pairs of values for such a function may be tabulated as under :

x	y
$-2 \leq x < -1$	–2
$-1 \leq x < 0$	–1
$0 \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2
$3 \leq x < 4$	3

With the above ordered pairs of values, the required graph of the *step* function would be drawn as under :

Graphic representation of the step function $y = [x]$

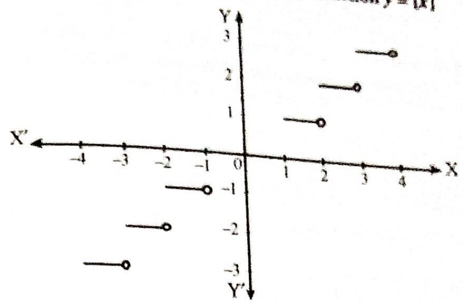


ILLUSTRATION 23. Draw a graph of the following function :

$$y = \log_a x$$

SOLUTION

The function is a logarithmic one. The required ordered pairs of values for the given logarithmic function may be tabulated as under :

x	1	a	a^2	a^3	a^4	a^5
y	0	1	2	3	4	5

With the above pairs of values, the required graph of the function would be drawn as follows :

Graphic representation of the logarithmic function $y = \log_a x$

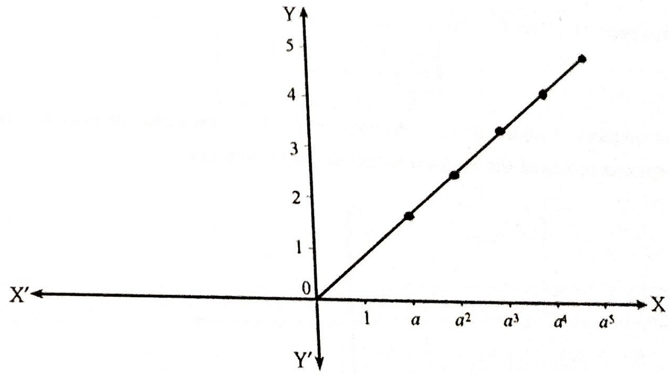


ILLUSTRATION 24. Draw the graph of the following function,

$$y = x^2 + 2x + 3, x \geq 0.$$

SOLUTION

The given function is a quadratic one. The ordered pairs of values for such a function may be tabulated as under :

x	0	1	2	3	4	5
y	3	6	11	18	27	38

With the above pairs of values the required graph will be drawn as follows :

Graphic representation of the quadratic function

$f(x) = x^2 + 2x + 3, x \geq 0$

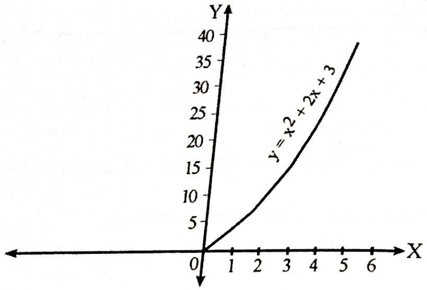


ILLUSTRATION 25. Represent the function $f(x) = sgn(x)$ graphically.

SOLUTION

The given function is a signum one which is expressed as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

The ordered pairs of values for such a function may be tabulated as under :

x	>0	0	<0
y	1	0	-1

With the above pairs of values the required graph of the function would be drawn as under :

Graphic representation of the Signum function, $f(x) = \text{Sgn}(x)$

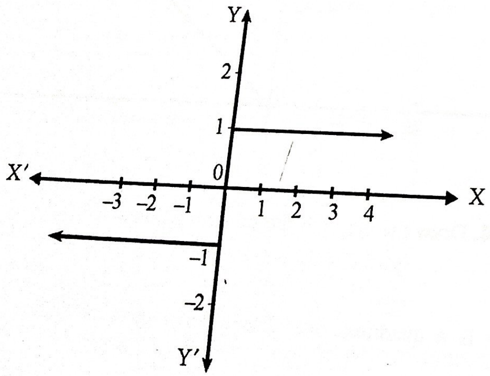


ILLUSTRATION 26. Represent the function $f(x) = \frac{1}{x}$ through a graph given $x \in R$, and $x \neq 0$.

SOLUTION

The given function is not defined at $x = 0$.

Thus, the ordered pairs of the values for the positive values of x for such a function may be tabulated as under :

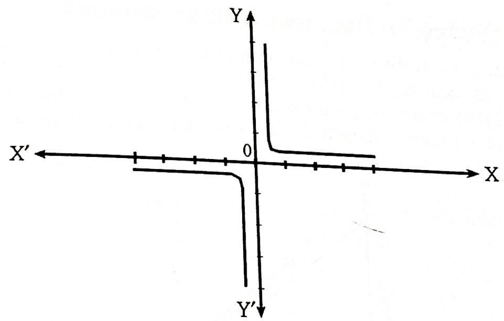
x	0·01	0·1	1	2	3	4	5	6	7	10	20	100	1000
y	100	10	1	0·5	0·33	0·25	0·20	0·16	0·14	0·1	0·05	0·01	0·001

And the ordered pairs of values for the negative values of x for such a function may be tabulated as under :

x	–0·01	–0·1	–1	–2	–3	–4	–5	–10	–20	–100	–1000
y	–100	–10	–1	–5	–0·33	–0·25	–0·20	–0·1	–0·05	–0·01	–0·001

With the above pairs of values, the graph of the function would appear as under :

Graphic representation of the function, $f(x) = \frac{1}{x}$.



The above graph of the function $y = \frac{1}{x}$ is called the rectangular hyperbola.

EXERCISE (C)

1. Represent the following functions by means of the appropriate graphs :

(i) $f(x) = 3 + 2x$ (ii) $y = x^2 + 5$

(iii) $y = \frac{3+2x}{2}$ (iv) $y = \sqrt{x}$

(v) $y = \frac{x}{x-1}, x \neq 1$ (vi) $y = \log_a^x$

2. Make Graphical representations of the following functions :

(i) $f(x)=5^x$ (ii) $f(x)=3^{3x}$

(iii) $f(x)=\log x^2$ (iv) $f(x)=x^3$

(v) $f(x)=x^2-2x+3$

3. Draw graphs for the following functions :

(i) $f(x)=|x|$ (ii) $f(x)=\frac{x}{2}+1$

4. Draw graph of the following function :

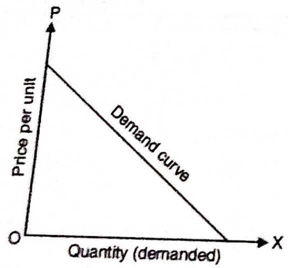
$$f(x)=\begin{cases}3+2x, & \text{when } x\le 0\\ 3-2x, & \text{when } x>0\end{cases}$$

5. Draw the graph of the following function :

$$f(x)=\begin{cases}x-1, & \text{when } x>0\\ -1/2, & \text{when } x=0\\ x+1, & \text{when } x<0\end{cases}$$

6. Functions Relating To Business And Economics

The concept of function is really a very natural one for many areas of applications. For instance, w the economist says that the quantity demanded is a function of a price, he means that price and quantity related in such a way that to each price (in some domain of prices), there corresponds exactly one quan demanded. In this section we present some functions which are very useful in business and economics



Demand Function. For each price level of a product, there is a corresponding quantity of that prod that consumers will demand during some time period. An equation that relates price per unit and quan demanded at that price is called a *demand function*. It graph is called a *demand curve*. If p is the price unit of a certain product and x is the number of units of that product that consumers will demand dur some time period at that price, then we can express the demand function as

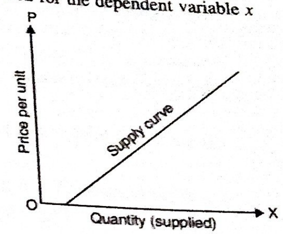
$$x = f(p)$$

Here x is the dependent variable and p is the independent variable. Since negative prices and quant are meaningless, both x and p must be non-negative.

Usually, an increase in price corresponds to a decrease in given above quantity demanded and v versa. The figure shows a demand curve which is a straight line. It is called a *linear* demand curve.

The reader should note that in the said figure the horizontal axis is used for the dependent variable x and the vertical axis for the independent variable p. The purpose is to be consistent with the convention that most economists use.

Supply Function. In response to various prices, there is a corresponding quantity of product that producers are willing to supply to the market during some time period. An equation that relates price per unit and the quantity supplied at that price is called a *supply function* and its graph is called a *supply curve*.



If p denotes the price per unit and x denotes the corresponding quantity supplied, then the supply function can be expressed as

$$x = g(p)$$

As before x and p are non-negative. Usually, an increase in price corresponds to an increase in the quantity supplied and a decrease in the price brings about a decrease in supply. The figure given above shows a linear supply curve.

Cost Function. Let C be the total cost incurred in the production of x units of a commodity. Then a function, say,

$$C = C(x)$$

relating C and x is called a *cost* function and its graph is called a *cost* curve. It may be noted that Total cost = Fixed cost + Variable cost,

Where *fixed cost* (*or overhead*) is the sum of all costs that are independent of the level of production, such as rent, insurance etc. and *variable cost* is the sum of all cost that are dependent on the level of production, such as cost of material, labour, etc.

Revenue Function. Let R be the total revenue or income the company makes by selling x units of a product at price p per unit. Then R is given by the formula

$$R = px$$

R is called the *total revenue* function.

Profit function. If $R(x)$ and $C(x)$ be the total revenue received and the total cost incurred respectively in the production of x units of a product, then the function P given by

$$P(x) = R(x) - C(x)$$

is called the *profit* function.

Break even point. The break-even point is the level of production where the revenue from the sales is equal to the cost of production. At the break-even point, the company is neither making a profit nor losing money. In other words, profit is equal to 0 at this point. Graphically, the break-even point is the intersection of the total revenue and the total cost curves.

Consumption Function. If I denotes the total national income and C the total national consumption, then the function

$$C = f(I)$$

relating I and C is called the *consumption* function. The difference between I and consumption C is savings S. Thus $S = I - C$.

ILLUSTRATION 27. A publishing house finds that the cost of production directly attributed to each book is ₹ 30 and that the fixed costs are ₹ 15,000. If each book can be sold for ₹ 45, then determine :

(i) the cost function, (ii) the revenue function,

(iii) the profit function, and (iv) the break-even point

SOLUTION

(*i*) Let the number of books published by the publishing house be x. From the given information,
Variable cost = $30x$ and fixed cost = ₹ 15,000

Hence the total cost function, $C(x)$, may be written as

$$C(x) = 30x + 15{,}000.$$

(ii) Since each book is sold for ₹ 45, the revenue function, R(x), is given by

$$R(x) = 45x$$

(*iii*) Profit function, $P(x)$, is given by

$$P(x) = R(x) - C(x) = 45x - 30x - 15{,}000 = 15x - 15{,}000.$$

(*iv*) At the brak-even point, $P(x) = 0$. That is,

$$15x - 15{,}000 = 0 \text{ i.e., } x = 1{,}000$$

Hence 1,000 books should be produced and sold to achieve break- even point.

ILLUSTRATION 28. A profit making company wants to launch a new product. It observes that the fixed cost of the new product is ₹ 35,000 and the variable cost per unit is ₹ 500. The revenue function for the sale of x units is given by $5000x - 100x^2$. Find (i) profit function, (ii) break-even values, and (iii) the values of x that result in a loss.

SOLUTION

(*i*) If $R(x)$, $C(x)$, and $P(x)$ denote, respectively, the revenue function, the cost function, and the profit function, then we have

$$R(x) = 5{,}000 - 100x^2$$
$$C(x) = 35{,}000 + 500x$$
$$P(x) = R(x) - C(x) = 5{,}000x - 100x^2 - 35{,}000 - 500x = -100x^2 + 4500x - 35{,}000$$

Hence, the profit function is $P(x) = -100x^2 + 4500x - 35{,}000.$

(*ii*) For the break-even values, we have

$$P(x) = 0$$

i.e., $-100x^2 + 4500x - 35{,}000 = 0$

$\Rightarrow$ $x^2 - 45x + 350 = 0$

Or $(x - 10)(x - 35) = 0$

$\Rightarrow$ $x = 10$ or $x = 35.$

Hence, the break-even values are 10 and 35.

(*iii*) To find the values of x that result in loss, we have

$$P(x) < 0$$

i.e., $-100x^2 + 4500x - 3500 < 0$

$\Rightarrow$ $x^2 - 45x + 350 > 0$

Or $(x - 10)(x - 35) > 0$

This is possible if either $x < 10$ or $x > 35$.

ILLUSTRATION 29. A calculator manufacturing company introduces production bonus to workers that increases the cost of the calculator. The daily cost of production C for x calculators is given by $C(x) = 205x + 55{,}000$.

(i) If each calculator is sold for ₹ 300, determine the minimum number that must be produced and sold daily to ensure no loss.

(ii) If the selling price is increased by ₹ 30 per piece, what would be the break-even point ?

(iii) If it is known that at least 500 calculators can be sold daily, what price the company should charge per piece of calculator to guarantee no loss ?

SOLUTION

(*i*) Since the calculator is sold for ₹ 300, the revenue function, $R(x)$, equals

$$R(x) = 300x$$

The cost function is given by

$$C(x) = 205x + 55{,}000$$

At the break-even point, we have

$$R(x) = C(x)$$

i.e., $300x = 205x + 55{,}000$

$\Rightarrow$ $95x = 55{,}000$

or, $x = 578.95$

Thus at least 579 calculators must be produced and sold daily to ensure no loss.

(*ii*) If the selling price is increased by ₹ 30 per piece, then

$R(x) = 330x$

And hence, in this case, the break-even point is given by

$$330x = 205x + 55{,}000$$

i.e., $125x = 55{,}000$ or $x = 440$

Thus, with the increased selling price, break-even point is achieved when 440 calculators are produced.

(*iii*) If at least 500 calculators can be sold daily, the price 'p' per piece of calculator needed to ensure no loss is given by

$$500p = 205 \times 500 + 55{,}000 = 1{,}75{,}500$$

$$\Rightarrow \quad p = \frac{1{,}57{,}500}{500} = ₹\,315.$$

EXERCISE (D)

1. A shopkeeper charges ₹ 25 per item for purchasing 20 or less items. He gives some rebate if items bought are more. If the items bought are 50 or less, then a rebate of Re. 1/- per item and for purchase of more than 50 items rebate of ₹ 2 per item. Find the cost function.

$$\textbf{Ans. } C(x) = \begin{cases} 25x & x \le 20 \\ 24x & 20 < 5 \le 50 \\ 23x & 5 > 50 \end{cases}$$

2. Determine the numbers a, b, c such that the function $y = ax^2 + bx + c$ fits to the data points (–1, 7), (0, 4) and (2, 6). Hence express the quadratic function.

Ans. $a = 4\frac{4}{3}, b = -\frac{5}{3} c = 4$ and $y = \frac{4}{3}x^2 - \frac{5}{3}x + 4$

3. A garment manufacturer is planning production of new variety of shirts. It involves initially fixed cost of ₹ 1.5 lacs and a variable cost of ₹ 150 for producing each shirt.
If each shirt can be sold at ₹ 350, then find.
(*i*) the cost function, (*ii*) the revenue function,
(*iii*) the profit function, and (*iv*) the break-even point.
Ans. (*i*) $C(x) = 1,50,000 + 5x$ (*ii*) $R(x) = 350\,x$ (*iii*) $P(x = 200x - 1,50,000$ (*iv*) $= 750]$

4. For a new product, a manufacturer sets up an infrastructure which costs him ₹ 1,50,000. The variable cost (labour, material etc.) is estimated as ₹ 125 for each unit of the product. The price per unit is fixed at ₹ 160. Write down the Cost function C(*x*), Revenue function R(*x*) and Profit function P(*x*) for *x* units of the product. How many minimum number of units are to be produced in the first year of production so that there may be no loss during that year ?
Ans. (*i*) $C(x) = 1,50,000 + 125x$ (*ii*) $R(x) = 160x$ (*iii*) $P(x) = 35x - 1,50,000$ (*iv*) 42

5. A book publisher finds that the production cost of a book is ₹ 30 and the fixed cost per year amounts to ₹ 25,000. If each is sold at the rate of ₹ 50. Find
(i) Cost Function
(ii) Revenue Function
(iii) Minimum Number of books to be sold per year in order that there is no loss.
(**Ans.** (i) $C(x) = 25,000 + 30x$, (ii) $R(x) = 50x$, (iii) 1,25

6. The daily cost of production C for *x* units of an assembly is given by
$C(x) = ₹\, 12.5x + 6,400$.
(i) If each unit is sold ₹ 25, determine the minimum number of units that should be produced and sold to ensure no loss.
(ii) If the selling price is reduced by ₹ 2.5 per unit, what should be the Break-even point ?
(iii) If it is known that 500 units can be sold daily, what price per unit should be charged to guarantee no loss ?
(**Ans.** (i) 512 units, (ii) 640 units, (iii) ₹ 25.

□□□

Chapter
4

LIMITS AND CONTINUITY

1. LIMIT

Meaning

Limit, in the present context means a constant value to which the function of a variable, say $f(x)$ approaches as the variable, '*x*' approaches a given value say '*a*'. Thus, ***the limit of a function, say 'L', is that value to which f(x) approaches as the variable 'x' approaches a given value, 'a'***. The function, $f(x)$ approaches the fixed constant, 'L' in such a manner that the difference between the function and the constant *i.e.* $[f(x) - L]$, can be made smaller and smaller than any smallest possible positive number, say '$\in$' (read as epsilon) as we may choose, and this difference continues to remain less than the chosen positive number, '$\in$', so long the variable, '*x*' continues to appear nearer to '*a*', the particular value chosen for it. This means that as the variable, '*x*' approaches closer and closer to '*a*' the function of '*x*' $f(x)$ approaches closer and closer to 'L', so that δ, (the difference between $f(x)$ and L) continues to remain less than '$\in$', the chosen positive number, however small *i.e.* $\delta < \in$ when $x \to a$ and $f(x) \to L$.

Example. Let $f(x) = 2x + 5$ and $x \to 2$ (read as *x* tends to 2). Substituting the value 2 in place of '*x*' we get the limit of $f(x)$ or $L = 9$ (*i.e.*, $2(2) + 5$). Thus, the function approaches the limit, 9 as *x* approaches *a*, the value 2. Symbolically this is represented as under :

$$\lim_{x \to 2} (2x + 5) = 9$$

The above limit of the given function, $(2x + 5)$ can be proved from both of its lower and upper sides as follows :

(*i*) From the lower side

When *x* approaches	$f(x) = 2x + 5$	Limit	$f(x) - L\ (\delta)$
1.9	8.8	9	–0.2
1.99	8.98	9	–0.02
1.999	8.998	9	–0.002
1.9999	8.9998	9	–0.0002
and so on.			

In all the above cases, it must be seen that the differences between $f(x)$ and L is less than a positive number, however small.

(ii) From the upper side

When x approaches	$f(x) = 2x + 5$	Limit	$f(x) - L\ (\delta)$
2.1	9.2	9	0.2
2.01	9.02	9	0.02
2.001	9.002	9	0.002
2.0001	9.0002	9	0.0002
and so on.			

Notes : 1. From the above table it must be seen that as 'x' moves closer and closer to 2, $f(x)$ mov closer and closer to the limit 9.

2. In the above case, the smallest possible positive number, $\in$ (epsilon) to be chosen must be more than the value of δ.

3. The limit of the function, as 'x' approaches 'a' is not the value of the function when $x = a$. In the above example, the limit of the function is 9, whereas, the value of the function is indeterminate or indefinite.

Definition of Limit

From the foregoing discussion the 'limit' with reference to a function may be defined as under :

"L is said to be the limit of the function, $f(x)$ as x approaches 'a', if the difference between L and $f(x)$ can be made as small as we can by taking 'x' sufficiently nearer to 'a' and is denoted symbolically as $\lim\limits_{x \to a} f(x) = L$."

Characteristics

The essential characteristics of the limit of a function, may be outlined as under :

(i) The variable 'x' may assume any value sufficiently nearer to 'a' on both of its lower and upper sides but never equals with 'a'.

(ii) The quantity $\lim\limits_{x \to a} f(x)$ depends only on the values of $f(x)$ for 'x' near a but not for 'x' equal to a.

(iii) As 'x' gets closer and closer to 'a' the value of $f(x)$ might not approach any fixed number and in such a case we may say that $\lim\limits_{x \to a} f(x)$ does not exist.

(iv) The difference between the $f(x)$ and the limit always remains less than the smallest possible positive number as may be chosen.

(A) Algebra of Limits

1. Additive method

Under this method, the limit of the sum of any two or more functions say $f(x)$ and $g(x)$ is determined by the following model.

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

ILLUSTRATION 1. Find the limit of the function $x^2 - 4x + 3$ when $x \to 2$.

SOLUTION

In the fitness of the data, the limit of the function will be computed by the additive method as under :

$$\lim_{x \to 2} (x^2 - 4x + 3) = \lim_{x \to 2} (x^2) - \lim_{x \to 2} (4x) + \lim_{x \to 2} (3)$$

$$= 2^2 - (4 \times 2) + 3 = 4 - 8 + 3 = -1$$

Thus, the required limit of the function $= -1$

ILLUSTRATION 2. Evaluate the $\lim\limits_{x \to 0} (x^2 + 2x + 5)$.

SOLUTION

By the additive theorem we have,

$$\lim_{x \to 0} (x^2 + 2x + 5) = \lim_{x \to 0} x^2 + \lim_{x \to 0} 2x + \lim_{x \to 0} 5 = \lim_{x \to 0} x^2 + 2 \lim_{x \to 0} x + \lim_{x \to 0} 5$$

$$= 0^2 + (2 \times 0) + 5 = 0 + 0 + 5 = 5.$$

2. Multiplicative method

Under this method the limit of the product of any two or more functions say $f(x)$ and $f(g)$ is computed by the following model :

$$\lim_{x \to a} [f(x).\ g(x)] = \lim_{x \to a} f(x)\ .\ \lim_{x \to a} g(x)$$

ILLUSTRATION 3. Find the $\lim\limits_{x \to 3} (x-4)(x+5)$.

SOLUTION

By the multiplication theorem we have,

$$\lim_{x \to 3} [(x-4)(x+5)] = \lim_{x \to 3} (x-4).\ \lim_{x \to 3} (x+5)$$

Again, by the multiplicative theorem through additive theorem we have,

$$= \left[\lim_{x \to 3} x - \lim_{x \to 3} 4\right]\left[\lim_{x \to 3} x + \lim_{x \to 3} 5\right] = (3-4)(3+5) = -1 \times 8 = -8$$

ILLUSTRATION 4. Prove that $\lim\limits_{x \to a} x^n = a^n$, when n is a positive number.

SOLUTION

We know that $x^n = x\ .\ x\ .\ x \ldots (n$ times)

By the multiplicative theorem we have,

$$\lim_{x \to a} x^n = \lim_{x \to a} x.\ \lim_{x \to a} x.\ \lim_{x \to a} x \ldots (n \text{ times})$$

$$= a\ .\ a\ .\ a \ldots\ldots\ldots (n \text{ times}) = a^n \quad (\text{where } \lim_{x \to a} x = a)$$

3. Constantive method

Under this method the limit of the product of a constant, say 'C' and a function say $f(x)$ is determined by the following model :

$$\lim_{x \to a} C\ .\ f(x) = C \lim_{x \to a} f(x)$$

ILLUSTRATION 5. ***Find the limit 25 x^3 when $x \to 5$.***

SOLUTION

By the constantive model we have,

$$\lim_{x \to 5} (25x^3) = 25 \lim_{x \to 5} x^3 = 25 \times 5^3 = 25 \times 125 = 3125 \quad (\text{where } \lim_{x \to 5} x = 5).$$

ILLUSTRATION 6. Evaluate $5x^3 + 3x^2 - 2x$ when $x \to 2$.

SOLUTION

The given function involves both the constant and the sum operations. Hence the limit of the function will be determined by the constantive model through the additive model as under :

$$\lim_{x\to 2}(5x^3+3x^2-2x) = \lim_{x\to 2} 5x^3 + \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 2x = 5\lim_{x\to 2} x^3 + 3\lim_{x\to 2} x^2 - 2\lim_{x\to 2} x$$

$$= 5(2)^3 + 3(2)^2 - 2(2) = 5\times 8 + 3\times 4 - 4 = 48.$$

Thus, the required limit of the function = 48.

4. Dividing method

Under this method, the limit of the quotient of any two functions say $f(x)$ and $g(x)$ is determined by the following model :

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}} \quad [\text{where } g(x) \neq 0.]$$

ILLUSTRATION 7. Evaluate $\lim_{x\to 1}\left(\frac{x+9}{2x+5}\right)$.

SOLUTION

By the dividing model we have,

$$\lim_{x\to 1}\left(\frac{x+9}{2x+5}\right) = \frac{\lim_{x\to 1}(x+9)}{\lim_{x\to 1}(2x+5)} = \frac{1+9}{2\times 1+5} = \frac{10}{7}.$$

Hence, the required limit of the function is $\frac{10}{7}$.

5. Left hand side method

When 'x' approaches 'a' from the left hand side of a limit (*i.e.*, when $x \to a$ by gradually increasing through values less than 'a' *viz*: 1.9, 1.99, 1.999 etc. all of which < 2), the limit of the function is determined by the following model :

$$\boxed{\lim_{x\to a-} f(x) = \lim_{h\to 0} f(a-h) \text{ or } \lim_{h\to 0} f(a-0) \text{ or } \lim_{h\to 0} f(a^-)}$$

where, $h = a$ very small change in $f(x)$, equivalent to δx, *i.e.*, $a - x$ and $x = a - h$.

ILLUSTRATION 8. Find the $\lim_{x\to 0}\frac{1}{x}$ from the left hand side.

SOLUTION

By the formula of limit from L.H.S, we have

$$\lim_{x\to a-} f(x) = \lim_{h\to 0} f(a-h)$$

Thus, $\lim_{x\to 0-}\left(\frac{1}{x}\right) = \lim_{h\to 0}\left(\frac{1}{0-h}\right) = \lim_{h\to 0}\left(\frac{1}{-h}\right) = \lim_{h\to 0}\frac{-1}{h} = \frac{-1}{0} = -\infty$

ILLUSTRATION 9. Evaluate $\lim_{x\to 3}(x^2+3x)$ from the L.H.S.

SOLUTION

By the formula of limit from L.H.S. we have,

$$\lim_{x\to a-} f(x) = \lim_{h\to 0} f(a-h)$$

Thus, $\lim_{x\to a-}(x^2+3x) = \lim_{h\to 0}(3-h)^2 + 3\lim_{h\to 0}(3-h) = 9 + 3(3) = 18$

6. Right hand side method

When x approaches a from the right hand side *i.e* 'when x tends to a by gradually decreasing through values greater than 'a' *viz*., 2.1, 2.01, 2.001 etc. all of which are greater than 2, the limit of the function is determined by the following model :

$$\lim_{x\to a+} f(x) = \lim_{h\to 0} f(a+h); \text{ or } \lim_{h\to 0} f(a+0), \text{ or } \lim_{h\to 0} f(a+)$$

Where, $h = a$ very small change in $f(x)$ equivalent to

δ_x *i.e.*, $x - a$ and $x = (a+h)$.

Limit of a function exists where the left hand limit and the right hand limit both exist and they are equal to the common value of the limit.

ILLUSTRATION 10. Evaluate $\lim_{x\to 3} x^2 + 3x$ from the R.H.S.

SOLUTION

By the model of R.H.S we have,

$$\lim_{x\to a+} f(x) = \lim_{h\to 0} f(a+h)$$

Thus $\lim_{x\to 3+}(x^2+3x) = \lim_{h\to 0}(3+0)^2 + 3\lim_{h\to 0}(3+0) = 9 + 3(3) = 18$

ILLUSTRATION 11. Show that $\lim_{x\to 0}\frac{1}{x}$ does not exist.

SOLUTION

$$\lim_{x\to 0^+}\frac{1}{x} = \lim_{h\to 0}\frac{1}{(0+h)} = \lim_{h\to 0}\frac{1}{h} = \infty$$

And, $\lim_{x\to 0^-}\frac{1}{x} = \lim_{h\to 0}\frac{1}{(0-h)} = \lim_{h\to 0}\frac{1}{-h} = -\lim_{h\to 0}\frac{1}{h} = -\infty$

Thus, $\lim_{x\to 0^+}\frac{1}{x} \neq \lim_{x\to 0^-}\frac{1}{x}$

Hence, $\lim_{x\to 0}\frac{1}{x}$ does not exist.

ILLUSTRATION 12. Show that $\lim_{x\to 1}[x]$ does not exist.

SOLUTION

We have, $\lim_{x\to 1}[x] = \lim_{h\to 0}[1+h] = 1 \quad \{\because [1+h] = 1\}$

Now, $\lim\limits_{x\to1}\dfrac{x^3-1}{x-1}=\lim\limits_{x\to1}\dfrac{(x-1)(x^2+x+1)}{(x-1)}=\lim\limits_{x\to1}x^2+x+1=1^2+1+1=3$

(*iii*) when $x=2$, the expression $\dfrac{x^3-3x^2+4}{x^4-8x^2+16}$ assumes an indeterminate form $\dfrac{0}{0}$.

$$\text{Now, }\lim_{x\to2}\frac{x^3-3x^2+4}{x^2-8x^2+16}=\lim_{x\to2}\frac{(x-2)(x^2-x-2)}{(x^2-4)^2}=\lim_{x\to2}\frac{(x-2)(x-2)(x+1)}{(x-2)^2(x+2)^2}$$

$$=\lim_{x\to2}\frac{x+1}{(x+2)^2}=\frac{2+1}{(2+2)^2}=\frac{3}{16}$$

(*iv*) When $x=1$, the expression $\dfrac{1}{x^2+x-2}-\dfrac{x}{x^3-1}$ assumes the indeterminate form $\infty-\infty$. So, we need simplification of the given expression as follows :

$$\lim_{x\to1}\left\{\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right\}\qquad(\infty-\infty\text{ form})$$

$$=\lim_{x\to1}\left\{\frac{1}{(x+2)(x-1)}-\frac{x}{(x-1)(x^2+x+1)}\right\}(\infty-\infty\text{ form})$$

$$=\lim_{x\to1}\left[\frac{x^2+x+1-x(x+2)}{(x+2)(x-1)(x^2+x+1)}\right]\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to1}\frac{(x^2+x+1-x^22-x)}{(x+2)(x-1)(x^2+x+1)}=\lim_{x\to1}\frac{-(x-1)}{(x+2)(x-1)(x^2+x+1)}$$

$$=\lim_{x\to1}\frac{-1}{(x+2)(x^2+x+1)}=\frac{-1}{(1+2)(1^2+1+1)}=-\frac{1}{9}$$

III. Rationalisation Method

This is particularly used when either the numerator, or the denominator, or both involve expression consisting of square roots and upon substitution of x, the rational expression takes the form $\dfrac{0}{0},\dfrac{\infty}{\infty}$ etc.

ILLUSTRATION 16. Evaluate the following limits :

(i) $\lim\limits_{x\to0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (ii) $\lim\limits_{x\to0}\dfrac{x}{\sqrt{a+x}-\sqrt{a-x}}$

(iii) $\lim\limits_{x\to4}\dfrac{x^2-16}{\sqrt{x^2+9}-5}$ (iv) $\lim\limits_{x\to0}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

SOLUTION

(*i*) When $x=0$, the expression $\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ takes the form $\dfrac{0}{0}$.

$$\text{Thus, }\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\lim_{x\to0}\frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})}$$

$$=\lim_{x\to0}\frac{2+x-2}{x\sqrt{2+x}+\sqrt{2}}=\lim_{x\to0}\frac{x}{x\sqrt{2+x}+\sqrt{2}}=\lim_{x\to0}\frac{1}{\sqrt{2+x}+\sqrt{2}}$$

$$=\frac{1}{\sqrt{2+0}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

(*ii*) When $x=0$, the expression $\dfrac{x}{\sqrt{a+x}-\sqrt{a-x}}$ takes the form $\dfrac{0}{0}$.

$$\text{Thus, }\lim_{x\to0}\frac{x}{\sqrt{a+x}-\sqrt{a-x}}=\lim_{x\to0}\frac{x(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$$

$$=\lim_{x\to0}\frac{x\sqrt{a+x}+\sqrt{a-x}}{(a+x)-(a+x)}=\lim_{x\to0}\frac{x\sqrt{a+x}+\sqrt{a-x}}{2x}$$

$$=\lim_{x\to0}\frac{\sqrt{a+x}+\sqrt{a-x}}{2}=\frac{\sqrt{a+0}+\sqrt{a-0}}{2}=\frac{2\sqrt{a}}{2}=\sqrt{a}$$

(*iii*) When $x=4$, the expression $\dfrac{x^2-16}{\sqrt{x^2+9}-5}$ takes the form $\dfrac{0}{0}$.

$$\text{Thus, }\lim_{x\to4}\frac{x^2-16}{\sqrt{x^2+9}-5}=\lim_{x\to4}\frac{(x^2-16)\left(\sqrt{x^2+9}+5\right)}{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}$$

$$=\lim_{x\to4}\frac{(x^2-16)\sqrt{x^2+9}+5}{x^2+9-25}\quad\lim_{x\to4}\frac{(x^2-16)\left(\sqrt{x^2+9}+5\right)}{(x^2-16)}$$

$$=\lim_{x\to4}\sqrt{x^2+9}+5=\sqrt{(4)^2+9}+5=5+5=10$$

(*iv*) When $x=a$, the expression $\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ assumes the form $\dfrac{0}{0}$.

$$\text{Thus, }\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}=\lim_{x\to a}\frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}-2\sqrt{x}\right)}$$

$$=\lim_{x\to a}\frac{(a+2x-3x)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{(3a+x-4x)\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$=\lim_{x\to a}\frac{(a-x)\sqrt{3a+x}+2\sqrt{x}}{3(a-x)\sqrt{a+2x}+\sqrt{3x}}=\lim_{x\to a}\frac{\sqrt{3a+x}+2\sqrt{x}}{3\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$=\frac{\sqrt{3a+a}+2\sqrt{a}}{3\left(\sqrt{a+2a}+\sqrt{3a}\right)}=\frac{4\sqrt{a}}{3.2\sqrt{3a}}=\frac{4\sqrt{a}}{3.2\sqrt{3}.\sqrt{a}}=\frac{2}{3\sqrt{3}}$$

Now, $\lim\limits_{x\to1}\frac{x^3-1}{x-1} = \lim\limits_{x\to1}\frac{(x-1)(x^2+x+1)}{(x-1)} = \lim\limits_{x\to1} x^2+x+1 = 1^2+1+1=3$

(*iii*) when $x=2$, the expression $\frac{x^3-3x^2+4}{x^4-8x^2+16}$ assumes an indeterminate form $\frac{0}{0}$.

Now, $\lim\limits_{x\to2}\frac{x^3-3x^2+4}{x^2-8x^2+16} = \lim\limits_{x\to2}\frac{(x-2)(x^2-x-2)}{(x^2-4)^2} = \lim\limits_{x\to2}\frac{(x-2)(x-2)(x+1)}{(x-2)^2(x+2)^2}$

$= \lim\limits_{x\to2}\frac{x+1}{(x+2)^2} = \frac{2+1}{(2+2)^2} = \frac{3}{16}$

(*iv*) When $x=1$, the expression $\frac{1}{x^2+x-2}-\frac{x}{x^3-1}$ assumes the indeterminate form $\infty-\infty$. So, need simplification of the given expression as follows :

$$\lim_{x\to1}\left\{\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right\} \quad (\infty-\infty \text{ fo}$$

$$= \lim_{x\to1}\left\{\frac{1}{(x+2)(x-1)}-\frac{x}{(x-1)(x^2+x+1)}\right\} (\infty-\infty \text{ form})$$

$$= \lim_{x\to1}\left[\frac{x^2+x+1-x(x+2)}{(x+2)(x-1)(x^2+x+1)}\right]\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to1}\frac{(x^2+x+1-x^2 2-x)}{(x+2)(x-1)(x^2+x+1)} = \lim_{x\to1}\frac{-(x-1)}{(x+2)(x-1)(x^2+x+1)}$$

$$= \lim_{x\to1}\frac{-1}{(x+2)(x^2+x+1)} = \frac{-1}{(1+2)(1^2+1+1)} = -\frac{1}{9}$$

III. Rationalisation Method

This is particularly used when either the numerator, or the denominator, or both involve expres consisting of square roots and upon substitution of x, the rational expression takes the form $\frac{0}{0}, \frac{\infty}{\infty}$ etc.

ILLUSTRATION 16. Evaluate the following limits :

(i) $\lim\limits_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$ (ii) $\lim\limits_{x\to0}\frac{x}{\sqrt{a+x}-\sqrt{a-x}}$

(iii) $\lim\limits_{x\to4}\frac{x^2-16}{\sqrt{x^2+9}-5}$ (iv) $\lim\limits_{x\to0}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

SOLUTION

(*i*) When $x=0$, the expression $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ takes the form $\frac{0}{0}$.

Thus, $\lim\limits_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim\limits_{x\to0}\frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})}$

$$= \lim_{x\to0}\frac{2+x-2}{x\sqrt{2+x}+\sqrt{2}} = \lim_{x\to0}\frac{x}{x\sqrt{2+x}+\sqrt{2}} = \lim_{x\to0}\frac{1}{\sqrt{2+x}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2+0}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(*ii*) When $x=0$, the expression $\frac{x}{\sqrt{a+x}-\sqrt{a-x}}$ takes the form $\frac{0}{0}$.

Thus, $\lim\limits_{x\to0}\frac{x}{\sqrt{a+x}-\sqrt{a-x}} = \lim\limits_{x\to0}\frac{x(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$

$$= \lim_{x\to0}\frac{x\sqrt{a+x}+\sqrt{a-x}}{(a+x)-(a+x)} = \lim_{x\to0}\frac{x\sqrt{a+x}+\sqrt{a-x}}{2x}$$

$$= \lim_{x\to0}\frac{\sqrt{a+x}+\sqrt{a-x}}{2} = \frac{\sqrt{a+0}+\sqrt{a-0}}{2} = \frac{2\sqrt{a}}{2} = \sqrt{a}$$

(*iii*) When $x=4$, the expression $\frac{x^2-16}{\sqrt{x^2+9}-5}$ takes the form $\frac{0}{0}$.

Thus, $\lim\limits_{x\to4}\frac{x^2-16}{\sqrt{x^2+9}-5} = \lim\limits_{x\to4}\frac{(x^2-16)\left(\sqrt{x^2+9}+5\right)}{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}$

$$= \lim_{x\to4}\frac{(x^2-16)\sqrt{x^2+9}+5}{x^2+9-25} \quad \lim_{x\to4}\frac{(x^2-16)\left(\sqrt{x^2+9}+5\right)}{(x^2-16)}$$

$$= \lim_{x\to4}\sqrt{x^2+9}+5 = \sqrt{(4)^2+9}+5 = 5+5 = 10$$

(*iv*) When $x=a$, the expression $\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ assumes the form $\frac{0}{0}$.

Thus, $\lim\limits_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} = \lim\limits_{x\to a}\frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}-2\sqrt{x}\right)}$

$$= \lim_{x\to a}\frac{(a+2x-3x)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{(3a+x-4x)\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a}\frac{(a-x)\sqrt{3a+x}+2\sqrt{x}}{3(a-x)\sqrt{a+2x}+\sqrt{3x}} = \lim_{x\to a}\frac{\sqrt{3a+x}+2\sqrt{x}}{3\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3\left(\sqrt{a+2a}+\sqrt{3a}\right)} = \frac{4\sqrt{a}}{3.2\sqrt{3a}} = \frac{4\sqrt{a}}{3.2\sqrt{3}.\sqrt{a}} = \frac{2}{3\sqrt{3}}$$

IV. Evaluation of Algebraic limits when $x \to \infty$

To evaluate this type of limits we follow the following steps.

Step-I : Write down the given expression in the form of a rational function *i.e.*, $\dfrac{f(x)}{g(x)}$, If it is no so.

Step-II : If k is the highest power of x in the numerator and denominator both, then divide each term in the numerator and denominator by x^k.

Step-III : Use the result $\lim\limits_{x\to\infty} \dfrac{1}{x^n} = 0$ when $n > 0$.

The following illustrations would show how the above types of limits are evaluated.

ILLUSTRATION 17. Evaluate the following limits.

(i) $\lim\limits_{x\to\infty} \dfrac{ax^2+bx+c}{dx^2+cx+f}$ (ii) $\lim\limits_{x\to\infty} \left(\dfrac{2x^2+4x+7}{3x^2+6x+5}\right)$

(iii) $\lim\limits_{x\to\infty} \dfrac{5x-6}{\sqrt{4x^2+9}}$ (iv) $\lim\limits_{x\to\infty} \sqrt{x^2+x+1}-\sqrt{x^2+1}$

SOLUTION

(*i*) Here, the expression assumes the form $\dfrac{\infty}{\infty}$. We notice that the highest power of x in both the numerator and denominator is 2. So we divide each term in both numerator and denominator by x^2.

$$\therefore \lim_{x\to\infty} \frac{ax^2+bx+c}{dx^2+ex+f} = \lim_{x\to\infty} \frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}} = \frac{a+0+0}{d+0+0} = \frac{a}{d}$$

$$(ii)\ \lim_{x\to\infty} \frac{2x^2+4x+7}{-3x^2+6x+5} = \lim_{x\to\infty} \frac{2+\frac{4}{x}+\frac{7}{x^2}}{-3+\frac{6}{x}+\frac{5}{x^2}}$$

(By dividing each term in the numerator and denominator by x^2)

$$= \frac{2+0+0}{-3+0+0} = -\frac{2}{3}$$

$$(iii)\ \lim_{x\to\infty} \frac{5x-6}{\sqrt{4x^2+9}} = \lim_{x\to\infty} \frac{5-\frac{6}{x}}{\sqrt{4+\frac{9}{x^2}}} = \frac{5+0}{\sqrt{4+0}} = \frac{5}{2}$$

(*iv*) Here, the expression assumes the form $\infty - \infty$ as $x \to \infty$. So, we reduce the given expression to the rational form $\dfrac{f(x)}{g(x)}$.

We have $\lim\limits_{x\to\infty} \sqrt{x^2+x+1}-\sqrt{x^2+1}$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}$$

$$= \lim_{x\to\infty} \frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}}$$

(Dividing numerator and denominator by x, and x^2 respectively)

$$= \frac{1}{\sqrt{1+0+0}+\sqrt{1+0}} = \frac{1}{1+1} = \frac{1}{2}.$$

V. STANDARD LIMIT THEOREMS

The following are some of the important limits theorems which we need in differentiation of some functions.

Theorem I : $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}\ a>0$

Theorem II : $\lim\limits_{x\to 0} \dfrac{a^x-1}{x} = \log_e a$

Theorem III: $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = 1$

Theorem IV: $\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = 1$

Theorem V : $\lim\limits_{x\to 0} \left[\dfrac{(1+x)^n-1}{x}\right] = n$

Remember 1. $\lim\limits_{x\to 0} e^x = 1$ **2.** $\lim\limits_{x\to\infty} x^n = 0$ where $-1<x<1$

ILLUSTRATION 18. Evaluate the following limits :

(i) $\lim\limits_{x\to 9} \dfrac{x^{3/2}-27}{x-9}$ (ii) $\lim\limits_{x\to a} \dfrac{x^m-a^m}{x^n-a^n}$

(iii) $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{x}$ (iv) $\lim\limits_{x\to 0} \dfrac{e^x-1}{x}$

(v) $\lim\limits_{x\to 1} \dfrac{x-1}{\log_e x}$ (vi) $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-2}{x^2}$

SOLUTION

(i) when $x = 9$, the expression $\frac{x^{3/2}-27}{x-9}$ assumes the form $\frac{0}{0}$.

Now, $\lim_{x\to 9}\frac{x^{3/2}-27}{x-9} = \lim_{x\to 9}\frac{x^{3/2}-9^{3/2}}{x-9} = \frac{3}{2}\,9^{3/2-1} = \frac{3}{2}\times 3 = \frac{9}{2}$ $\quad\left(\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = n\,a^{n-1}\right)$

(ii) $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}$ is in the form $\frac{0}{0}$ when $x = 0$

$$= \lim_{x\to a}\left[\frac{x^m-a^m}{x-a}\times\frac{(x-a)}{x^n-a^n}\right]$$

$$= \lim_{x\to a}\left[\frac{x^m-a^m}{x-a}\div\frac{x^n-a^n}{x-a}\right] = \lim_{x\to a}\frac{x^m-a^m}{x-a}\div\lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$= m\,a^{m-1}\div n\,a^{n-1} = \frac{m}{n}\,a^{m-1-n+1} = \frac{m}{n}\,a^{m-n}$$

(iii) $\lim_{x\to 0}\frac{a^x-b^x}{x}$ $\quad\left(\text{i.e., in the form } \frac{0}{0}, \text{ when } x = 0\right)$

$$= \lim_{x\to 0}\frac{a^x-1+1-b^x}{x} = \lim_{x\to 0}\left[\frac{a^x-1}{x}-\frac{b^x-1}{x}\right]$$

$$= \lim_{x\to 0}\frac{a^x-1}{x} - \lim_{x\to 0}\frac{b^x-1}{x} = \log_e a - \log_e b = \log_e\left(\frac{a}{b}\right)$$

(iv) $\lim_{x\to 0}\frac{e^{-x}-1}{x}$, (Let, $-x = z$, as $x\to 0$, so $z\to 0$)

$$= \lim_{z\to 0}\frac{e^z-1}{-z} = -\lim_{z\to 0}\frac{e^z-1}{z} = -1. \quad\left(\because \lim_{x\to 0}\frac{e^x-1}{x} = 1\right)$$

(v) $\lim_{x\to 1}\frac{x-1}{\log_e x} = \lim_{h\to 0}\frac{1+h-1}{\log(1+h)} = \lim_{h\to 0}\frac{h}{\log(1+h)}$ (by using right hand method)

$$= \frac{1}{\lim_{h\to 0}\frac{\log(1+h)}{h}} = \frac{1}{1} = 1 \quad\left(\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$$

(vi) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2} = \lim_{x\to 0}\frac{e^{2x}+1-2e^x}{x^2.e^x}$ (multiplying e^x in both numerator and denominator)

$$= \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2\frac{1}{e^x} = \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2\cdot\frac{1}{\lim_{x\to 0}e^x} = 1^2\times 1 = 1$$

EXERCISE (A)

1. Show that :

(i) $\lim_{x\to 1}(7-3x) = 4$

(ii) $\lim_{x\to 3}(x^2+5) = 14$

(iii) $\lim_{x\to 0}\left(\frac{7}{x+7}\right) = 1$

(iv) $\lim_{x\to 3}\frac{x^2-9}{x-3} = 6$

(v) $\lim_{x\to 0}(5x^2-9x+9) = 9$

(vi) $\lim_{x\to 1}\frac{x^2-4x+3}{x^2-6x+5} = \frac{1}{2}$

2. Evaluate :

(i) $\lim_{x\to 3}\frac{x^4-81}{x-3}$

(ii) $\lim_{x\to 4}\frac{x^4-256}{x-4}$

(iii) $\lim_{x\to 0}\frac{x^2+2}{3x+4}$

(iv) $\lim_{x\to 1}\frac{x^2-1}{x-1}$

(v) $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

(vi) $\lim_{x\to 4}\frac{x^2-16}{x-4}$

(vii) $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

(viii) $\lim_{x\to 1}\frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$

Ans. (i) 108 (ii) 256 (iii) $\frac{1}{2}$ (iv) 2 (v) $\frac{1}{4}$
(vi) 8 (vii) 9/2 (viii) $-\frac{1}{10}$

3. Evaluate the following limits :

(i) $\lim_{x\to 1}(6x^2-4x+3)$

(ii) $\lim_{x\to 0}\{(x-2)^2+6\}$

(iii) $\lim_{x\to 7}\left(\frac{x^2-49}{x-7}\right)$

(iv) $\lim_{x\to 2}\frac{(x^2-6x+8)}{(x-2)}$

(v) $\lim_{x\to -5}\frac{(2x^2+9x-5)}{(x+5)}$

(vi) $\lim_{x\to 5}\frac{x^2-8x+15}{x-5}$

(vii) $\lim_{x\to 1}\frac{x^2+5x-6}{x^2-3x+2}$

(viii) $\lim_{x\to -2}\left(\frac{x^3+8}{x+2}\right)$

(ix) $\lim_{x\to 1}\frac{x^2+5x+16}{x-3x+2}$

(x) $\left\{\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}\right\}$

Ans. (i) 5 (ii) 10 (iii) 14 (iv) −2 (v) −11
(vi) 2 (vii) −7 (viii) 12 (ix) ∞ (x) $\frac{5}{3}(a+2)^{2/3}$

4. Prove that :

(i) $\lim_{x\to\infty} \frac{x-2}{x+1} = 1$

(ii) $\lim_{x\to\infty} \frac{5-2x^2}{3x+5x^2} = -\frac{2}{5}$

(iii) $\lim_{x\to\infty} \frac{3x^2+4x-1}{2x^2+x+2} = \frac{3}{2}$

(iv) $\lim_{x\to\infty} \frac{2x^2-3x+5}{3x^2-6} = \frac{2}{3}$

(v) $\lim_{x\to\infty} \frac{4-3x^2}{4x^2-x+5} = -\frac{3}{4}$

(vi) $\lim_{x\to\infty} \sqrt{x^2+2x+4} - x = 1$

(vii) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}-1} = 1$

(viii) $\lim_{x\to\infty} \frac{4x^2-5x+2}{\sqrt{x^4+2}} = 4$

5. Calculate the limits of the following functions :

(i) $\lim_{x\to 2} \frac{2x^2-7x+6}{5x^2-11x+2}$

(ii) $\lim_{x\to 0} \frac{4x^4+5x^3+7x^2+5x}{5x^2+8x^7+x}$

(iii) $\lim_{x\to 1} \frac{x^3-5x^2+2x+4}{x^3-3x^2+x-3}$

(iv) $\lim_{x\to -2} \frac{3-\sqrt{7+x}}{2+x}$

(v) $\lim_{x\to\infty} \frac{(3x+1)(2x-3)}{(x+2)(3x+7)}$

(vi) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

(vii) $\lim_{x\to 1} \frac{x-1}{2x^2-7x+5}$

(viii) $\lim_{x\to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

(ix) $\lim_{x\to 2} \frac{x^2-3x-2}{x^2+x-6}$

(x) $\lim_{x\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

Ans. (i) $\frac{1}{9}$ (ii) 5 (iii) 1 (iv) $\frac{1}{6}$ (v) 2

(vi) $\frac{1}{2}$ (vii) $-\frac{1}{3}$ (viii) 1 (ix) $\frac{1}{5}$ (x) $\frac{1}{2\sqrt{x}}$

6. Evaluate the following limits :

(i) $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

(ii) $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

(iii) $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)$

(iv) $\lim_{x\to 0} \frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$

(v) $\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x}$

(vi) $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1}$

(vii) $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x2}-\sqrt{x}}$

(viii) $\lim_{x\to 0} \frac{\log(1+x)}{x}$ **[Hint :** Expand log $(1+x)$]

(ix) $\lim_{x\to 0} \left(\frac{a^x-b^x}{a}\right)$ (**Hint :** Expand a^x and b^x)

(x) $\lim_{x\to 0} \left(\frac{e^x+e^{-x}-2}{x^2}\right)$ (**Hint :** Expand e^x and e^{-x})

Ans. (i) $\frac{1}{2}$ (ii) $\frac{1}{2\sqrt{2}}$ (iii) $-\frac{1}{2x^{3/2}}$ (iv) $2\sqrt{a}$ (v) $\frac{1}{2}$

(vi) 2 (vii) $\frac{2}{3\sqrt{3}}$ (viii) 1 (ix) $\log_e a - \log_e b$ (x) 1

7. Evaluate the following limits :

(i) $\lim_{x\to 3} \frac{x^2-x-6}{x^3-3x^2+x-3}$

(ii) $\lim_{x\to 2} \left(\frac{x}{x-2}-\frac{4}{x^2-2x}\right)$

(iii) $\lim_{x\to 0} \frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$

(iv) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

(v) $\lim_{x\to 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$

(vi) $\lim_{x\to 0} \frac{(1+x)^6-1}{(1+x)^2-1}$

(vii) $\lim_{x\to a} \frac{x^{5/7}-a^{5/7}}{x^{2/7}-a^{2/7}}$

(viii) $\lim_{x\to 0} \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$

(ix) $\lim_{x\to\infty} \frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}$

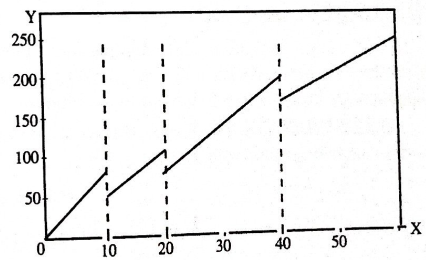
(x) $\lim_{x\to 0} \frac{5x^3-6}{\sqrt{9+4x^6}}$

Ans. (i) $\frac{1}{2}$ (ii) 2 (iii) $2\sqrt{a}$ (iv) $\frac{1}{2}$ (v) 2

(vi) 3 (vii) $\frac{5}{2}a^{3/7}$ (viii) 12 (ix) $\frac{3}{2}$ (x) $\frac{5}{2}$

2. CONTINUITY

Continuity in the present context relates to the continuity of a function at a point. A function $f(x)$ is said to be continuous at a point say at $x = a$, if corresponding to any arbitrarily assigned positive number, $\in$ (epsilon), however small (but not zero) there exists a positive number δ (*i.e.*, the difference between $f(x)$ and the limit of the function).



Graph of a discontinuous function

Geometrically speaking, a function, $f(x)$ is said to be continuous if its graph can be drawn at a stretch without lifting the pencil from the paper *i.e.*, without any gap, break or jump shown as under :

The points of discontinuity of a function are generally the gaps, breaks or jumps in the graph as shown above. The discontinuity of a function is very common in the field of trade and commerce where a trader may offer quantity discounts on larger

quantities purchased in order to attract voluminous orders as under :

Price in ₹ Quantity Ordered

5 for 10 units or less.

4 for more than 10 units but less than or equal to 20 units.

3.5 for more than 20 units but less than or equal to 30 units.

3 for more than 30 units.

If such data will be plotted on a graph paper, it will give a curve with breaks, as shown in the abo[ve] graph.

Definition of continuity

From the above discussion, the continuity with reference to a function may be defined as under :

"A function is said to be continuous at a point, $x = a$, if $f(x)$ possesses a finite and definite limit as x tends to the value, 'a' from either side (lower or upper side of a, say 3 *viz.* 2.9, 2.99 etc. or 3.1, 3.0[1] etc. respectively), **and each of these limits is equal to $f(a)$".**

Characteristics

From the above definition, the essential characteristics of a continuous function may be laid down a[s] follows :

(i) f is defined at a.

(ii) $\lim_{x\to a} f(x) = f(a)$.

(iii) $f(a)$, $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ all have finite and definite values.

(iv) The values of $\lim_{x\to a-} f(x)$, $f(a)$ and $\lim_{x\to a+} f(x)$ are identical so that

$$\lim_{h\to 0} f(a-h)) = f(a) = \lim_{h\to 0} f(a+h)$$

If, any of the above essential characteristics (known as conditions) is not satisfied, then the functio[n] will be deemed as **discontinuous** at the given point. In other words, the given point will be a point o[f] discontinuity of the function.

3. DIFFERENT METHODS OF DETERMINATION OF CONTINUITY

There are two different methods of determining the continuity or otherwise of a function. They are: (i) graphic method, and (*ii*) analytical method, detailed as under :

(i) Graphic method

Under this method the data relating to a function at a point or points are plotted on a graph paper. I[f] the curve so drawn shows no breaks throughout the interval, the function is said to be continuous. If there appears any break or gap in the curve the function is said to be discontinuous one.

ILLUSTRATION 19. The total cost of purchasing x units of an article is C and is proportional to [x] within each order interval given as under :

$$C = \begin{bmatrix} 2x; & 0 \le x \le 3000 \\ 15x; & 3000 < x \le 5000 \\ 10x; & x > 5000 \end{bmatrix}$$

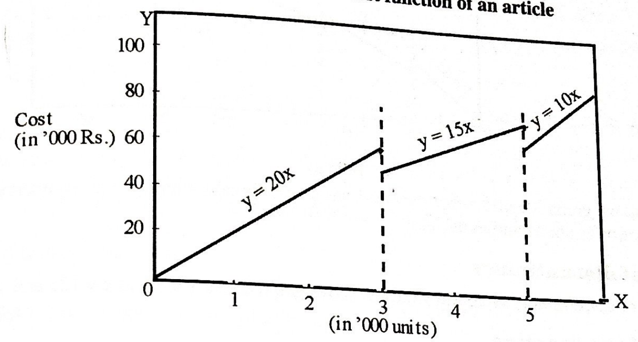
From the above information

(i) sketch the graph of the function, and (ii) find the points of discontinuity, if any.

SOLUTION

(*i*) Representing the cost by Y and quantities by X the graph of the cost function is drawn as under :

Graphic representation of the cost function of an article



From the above graph it appears that the curve has several breaks. Hence the cost function, thus given is not continuous.

(*ii*) The points of discontinuity as revealed from the above graph are : at $x = 3000$, and at $x = 5000$.

ILLUSTRATION 20. A firm offers the following price quotations for different volumes of orders for an article :

₹ 5 per unit for an orderfor 50 units or less.

₹ 4 per unit for an orderfor more than 50 units but not more than 125 units.

₹ 3.50 per unit for an orderfor more than 125 units but not more than 250 units.

₹ 3 per unitfor an order for more than 250 units.

(a) Determine by means of a graph, if the price function of the firm is continuous or not.

(b) Point out the breaking points of the function.

(c) Find the limit of the function as

$$x \to 20,\ x \to 50,\ x \to 125, \text{ and } x \to 250$$

(d) Explain why the function is discontinuous at

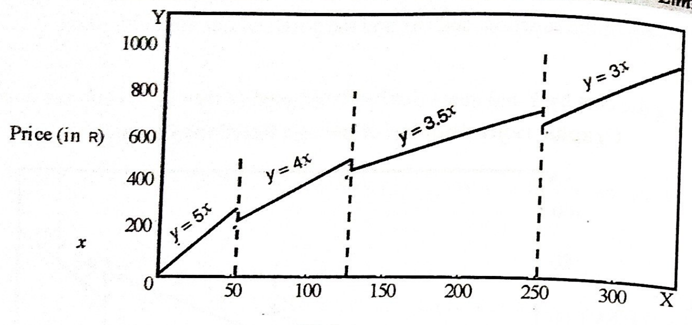
$$x = 50,\ x = 125 \text{ and } x = 250.$$

SOLUTION

(*a*) Representing the price by 'Y' and quantity by 'X', the price function of the firm is given by

$$Y = \begin{bmatrix} 5x & 0 \le x \le 50 \\ 4x & 50 \le x \le 125 \\ 3.5x & 125 \le x \le 250 \\ 3x & x > 254 \end{bmatrix}$$

Graphic representation of the price function



From the above graph it appears that the curve has several breaks within the given interval. Hence, the given price function is not a continuous one.

(a) Points of discontinuity

The points of discontinuity as appear from the above graph are, at $x = 50$, at $x = 125$ and at $x = 250$.

(b) Limits of the function

When $x \to 20$ limit is 100 (*i.e.* 20×5)

When $x \to 50$ left hand limit is 250 (*i.e.* 50×5)
right hand limit is 200 (*i.e.* 50×4)

When $x \to 125$ left hand limit is 500 (*i.e.* 125×4)
right hand limit is 437.50 (125×3.50)

When $x \to 250$ left hand limit is 875 (250×3.50)
right hand limit is 750 (250×3)

(*d*) The function is discontinuous at the points $x = 50$, $x = 125$, and $x = 250$, because the left hand and right hand limits at these points are not equal.

(*ii*) Analytical method

Under this method the continuity or otherwise of a function is judged by an analysis of the essential characteristics of a continuous function. A real function $f(x)$ is said to be the continuous at a point a if its domain, if $\lim_{x\to a} f(x)$ exists and equals $f(a)$. Thus, a function is continuous if it **satisfies the conditions that**

(*i*) **$f(a)$ is defined.** (*ii*) $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

If a function does not satisfies the above conditions, it is called as a discontinuous function.

ILLUSTRATION 21. ***Show that $f(x) = x^3$ is continuous at $x = 2$.***

SOLUTION

We have

(*i*) $f(a) = f(2) = (2)^3 = 8$

(*ii*) $\lim_{x\to 2^+} f(x) = \lim_{h\to a} (2+h)^3 = \lim_{h\to 0} (8 + h^3 + 12h + 6h^2) = 8$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to a} (2-h)^3 = \lim_{h\to 0} (8 - h^3 - 12h + 6h^2) = 8$$

$$\therefore \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$$

Hence, $f(x)$ is continuous at $x = 2$.

ILLUSTRATION 22. ***Show that $f(x) = [x]$ is not continuous at $x = n$, where n is an integer.***

SOLUTION

We have, $f(n) = [n] = n$;

$$\lim_{x\to n^+} f(x) = \lim_{h\to 0} f(n+h) = \lim_{h\to 0} [n+h] = n \qquad [\because [n+h] = n \text{ and } [n-h] = n-1]$$

$$\lim_{x\to n^-} f(x) = \lim_{h\to 0} f(n-h) = \lim_{h\to 0} (n-h) = n-1$$

Thus, $\lim_{x\to n^+} f(x) \neq \lim_{x\to n^-} f(x)$ and hence, $\lim_{x\to n} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = n$.

ILLUSTRATION 23. Show that the function $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION

It is given that $f(0) = 1$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{h}{|h|} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{-h}{|h|} = \lim_{h\to 0} \frac{-h}{h} = -1$$

$$\therefore \quad \lim_{x\to 0^+} f(x) \neq \lim_{h\to 0^-} f(x)$$

So, $\lim_{h\to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

ILLUSTRATION 24. If $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$

SOLUTION

Given that

$$f(1) = 2$$

$$\lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} (x+1) = \lim_{h\to 0} (1+h+1) = 2$$

$$\lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} (x+1) = \lim_{h\to 0} (1-h+1) = 2$$

Thus $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

ILLUSTRATION 25. Discuss the continuity of the function

$$f(x) = \begin{cases} 3x-2, \text{ when } x \le 0 \\ x+1, \text{ when } x > 0 \end{cases}$$

at $x = 0$.

SOLUTION

Clearly, $f(0) = 3\times 0 - 2 = -2$

$$\lim_{x\to 0^+} x+1 = \lim_{h\to 0} (0+h+1) = 1$$

$$\lim_{x\to 0} (3x-2) = \lim_{h\to 0} [3(0-h)-2] = \lim_{h\to 0} -3h-2 = -2$$

Thus, $\lim_{x\to 0^+} f(x) \ne \lim_{x\to 0^-} f(x)$ therefore $\lim_{x\to 0} f(x)$ does not exists.

Thus, the given function is discontinuous at $x = 0$.

ILLUSTRATION 26. Determine whether the following function is continuous or not at $x = 2$:

$$f(x) = x^2 - 4x + 3$$

SOLUTION

By the analytical method we are to investigate the characteri-stics of the function as follows :

(*i*) $f(a) = f(2) = 2^2 - 4(2) + 3 = -1$

$$\lim_{x\to 2^-} \left(x^2 - 4x + 3\right) = \lim_{h\to 0} (2-h)^2 - 4\lim_{h\to 0} (2-h) + \lim_{h\to 0} (3)$$

$$= \lim_{h\to 0} (2-h)^2 - 4\lim_{h\to 0} (2-h) + \lim_{h\to 0} (3)$$

$$= (2-0)^2 - 4(2-0) + 3 = 4 - 4(2) + 3 = -1$$

$$\lim_{x\to 2^+} (x^2 - 4x + 3) = \lim_{h\to 0} (2+h)^2 - 4\lim_{h\to 0} (2+h) + \lim_{h\to 0} (3)$$

$$= (2+0)^2 - 4(2+0) + 3 = -1$$

Thus, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

All the above three values are definite and identical as well. Hence, this characteristic is also satisfie

From the above analysis we find that the value of the function at $x = 2$, the left and right hand limits exist and are finite and identical as well.

Hence, the given function is continuous at the point $x = 2$.

It may be noted that the above function is continuous in any interval both for positive and nega values of x.

ILLUSTRATION 27. *Show that the $f(x) = 3x^2 + 4x - 5$ is continuous at $x = 3$. Also, prove that the f is continuous for all values of x.*

SOLUTION

Determination of continuity of the function by the analytical method.

(*a*) Text of continuity at $x = 3$

(*i*) $f(x) = 3x^2 + 4x - 5$

$$f(a) = 3(3)^2 + 4(3) - 5 = 34$$

Thus,

(*iii*) $\lim_{x\to 3^-} (3x^2 + 4x - 5) = \lim_{h\to 0} 3(a-h)^2 + 4\lim_{h\to 0} (a-h) - \lim_{h\to 0} (5)$

$$= 3(3-0)^2 + 4(3-0) - 5 = 3\times 9 + 4\times 3 - 5 = 34$$

$$\lim_{x\to 3^+} (3x^2 + 4x - 5) = \lim_{h\to 0} 3(a+h)^2 + 4\lim_{h\to 0} (a+h) - \lim_{h\to 0} (5)$$

$$= 3(3+0)^2 + 4(3+0) - 5 = 3\times 9 + 4\times 3 - 5 = 34$$

Thus, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

Hence the function $f(x)$ is continuous at $x = 3$.

(*b*) Test of continuity of $f(x)$ for all values of x

(*i*) Let $x = k$ be any value of x arbitrarily selected. Hence, $a = k$, and therefore,

$$f(k) = 3k^2 + 4k - 5$$

(*ii*) $\lim_{x\to k^-} (3x^2 + 4x - 5)$ ...(1)

$$= \lim_{h\to 0} \{3(k-h)^2 + 4(k-h) - 5\}$$

$$= \lim_{h\to 0} (3k^2 - 6kh + 3h^2 + 4k - 4h - 5) = 3k^2 + 4k - 5 \quad ...(2)$$

Further, $\lim_{x\to k^+} (3x^2 + 4x - 5)$

$$= \lim_{h\to 0} \{3(k+h)^2 + 4(k+h) - 5\}$$

$$= \lim_{h\to 0} (3k^2 + 6kh + 3h^2 + 4k + 4h - 5) = 3k^2 + 4k - 5 \quad ...(3)$$

Thus, $\lim_{x\to k^-} f(x) = \lim_{x\to k^+} f(x) = f(k)$

Since k is any arbitrary value of x it is therefore, held that $f(x)$ is continuous for all values of x.

ILLUSTRATION 28. Determine the continuity of the $\begin{cases} f(x) = \dfrac{|x|}{x}, x \ne 0 \\ f(x) = 0, x = 0 \end{cases}$

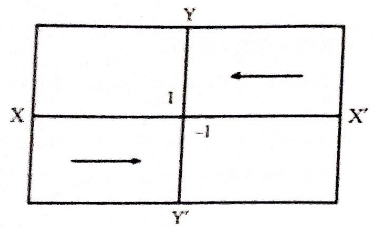
SOLUTION

Here, the $f(x)$ is given by

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

The above function of x can be graphically represented as under :

Graphic representation of the function



From the above graphic representation of the function it comes out that the function is discontinuous at $x = 0$.

By the analytical method

$$\lim_{x\to a-} f(x) = -1 \quad \text{and} \quad \lim_{x\to a+} f(x) = 1$$

Hence, analytically the function is discontinuous at this point as

$$\lim_{x\to a-} f(x) \neq \lim_{x\to a+} f(x)$$

ILLUSTRATION 29. Find if $f(x)$ is continuous at $x = 1$ when,

$$f(x) = \begin{Bmatrix} x-1, & 0 \le x < 1 \\ 1-x^2, & x > 1 \end{Bmatrix}.$$

SOLUTION

We know that under the analytical method, $f(x)$ will be continuous at $x = 1$ if,

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a-h) = f(a)$$

Hence, $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} [(1-(1+h)^2]$

$$= \lim_{h\to 0} \{1-(1+2h+h^2)\}$$

$$= \lim_{h\to 0} (1-1-2h-h^2) = \lim_{h\to 0} (-h^2-2h^2) = 0$$

$[\because$ at $x = 1+h, x > 1$, we have $f(x) = 1-x^2]$

Again $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{(1-h)-1\} = 0$

$[\because$ at $x = 1-h, x < 1$, we have $f(x) = (x-1)]$

Again, $f(a) = f(1) = 1 - 1 = 0$ $\qquad [\because$ at $x = 1, f(x) = 1-$

Hence, $f(a)$ is continuous at $x = 1$

ILLUSTRATION 30. State if the function is continuous at $x = 2$, when it is defined as follows:

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{when } 0 \le x < 2; \\ 2, & \text{when } x = 2; \\ x+1, & \text{when } x > 2; \end{cases}$$

SOLUTION

Here, $f(a)$, *i.e.*, $f(2) = 2$

This is a finite and a definite given number.

$$\lim_{x\to 2-} \frac{x^2-4}{x-2} = 4; \quad \lim_{x\to 2+} (x+1) = 3$$

The above results show that $\lim_{x\to 2^-} \neq \lim_{x\to 2^+} \neq f(a)$

Hence, the given function is not continuous at $x = 2$.

ILLUSTRATION 31. Find, if the function defined as follows is continuous at $x = 2$.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{for } x < 2 \\ 2, & \text{for } x \ge 2 \end{cases}$$

SOLUTION

Here, $f(x)$ or $f(a) = f(2) = 2$

$$\lim_{x\to 2-} \frac{x^2-4}{x-2} = 4; \quad \lim_{x\to 2+} 2 = 2$$

From the above it appears that $f(a) = f(2) = \lim_{x\to 2^+} \neq \lim_{x\to 2^-}$

Hence, the function given, is not continuous at $x = 2$.

ILLUSTRATION 32. Determine, if the function $f(x)$ defined as follows is continuous at $x = \frac{1}{2}$.

$$f(x) = \begin{cases} x, & \text{when } 0 \le x < \frac{1}{2} \\ 1, & \text{when } x = \frac{1}{2} \\ 1-x, & \text{when } \frac{1}{2} < x < 1 \end{cases}$$

SOLUTION

Here, $f(x)$ or $f(a) = \left(\frac{1}{2}\right) = 1$

This is a finite and a definite given number when $x = \frac{1}{2}$

$$\lim_{x\to \frac{1}{2}+} (x) = \lim_{h\to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2} [\because x \ge 0 \,\theta\, x < \frac{1}{2}]$$

$$\lim_{x\to \frac{1}{2}+} (1-x) = \lim_{h\to 0} \left\{1 - \left(\frac{1}{2} + h\right)\right\} = \frac{1}{2} [\because x > \frac{1}{2} \,\theta\, x < 1]$$

From the above results it follows that

$$f(a) = f\left(\frac{1}{2}\right) \neq \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^+} f(x)$$

Hence, the given function is not continuous at $x = \frac{1}{2}$.

4. CONTINUITY IN AN INTERVAL

The collection of all values lying between the two end points of a variable is called the continuity ... the interval of a function. The interval of a function may be either of the three types : *viz* : (*i*) closed interval, (*ii*) open interval and (*iii*) semi-open interval.

If both the end points '*a*' and '*b*' are included in the interval it is called the closed interval so that $a \leq$ ... $\leq b$. If both the end points are excluded from the interval it is called open interval so that $a < x < b$. If one of the end points alone is included in the interval, it is called semi-open interval so that $a \leq x < b$ or $a < x \leq b$.

A function, $f(x)$ is said to be continuous in the closed interval (a, b), if it is continuous for every value of x in $a < x < b$ and if $f(x)$ is continuous from the right at '*a*' and from the left at '*b*' *i.e* if $\lim_{x \to a^+} f(x)$ exists and is equal to $f(a)$ and $\lim_{x \to b^-} f(x)$ exists and is equal to $f(b)$.

The following illustrations show the continuity of a function in an interval.

ILLUSTRATION 33. Examine the continuity of the function defined as follows in the interval $(-3, 3)$

$$f(x) = \begin{cases} \dfrac{9x}{x+2}, & \text{where } x < 1 \\ 3, \text{ and } f(x) = \dfrac{x+3}{x}, & \text{where } x > 1 \end{cases}$$

SOLUTION

Here $f(x) = \frac{9x}{x+2}$ is to be considered for values of x lying between -3 and 1 because, this is the part of the function which is defined for value of $x < 1$. The denominator $(x + 2)$ becomes zero when $x = -2$ ... point between -3 and 1. This shows that the function is discontinuous at $x = -2$.

Again, $f(1) = 3$

$$\lim_{x \to 1^-} \frac{9x}{x+2} = \lim_{h \to 0} \frac{9(1-h)}{(1-h)+2} = 3$$

And

$$\lim_{x \to 1^+} \frac{x+3}{x} = \lim_{h \to 0} \frac{1+h+3}{1+h} = 4$$

Since the right hand limit 4, is not equal to the value of the function at $x = 1$, the given function is discontinuous at $x = 1$.

Hence, $f(x)$ is discontinuous at $x = 1$. However; for all other values of x it is continuous.

EXERCISE (B)

1. The cost function of a firm is given as below :

$$y = \begin{bmatrix} 30x, & 0 \leq x \leq 1000 \\ 26x, & 1000 < x \leq 5000 \\ 20x, & x > 5000 \end{bmatrix}$$

Where, x is the number of units ordered and y is the cost of purchasing x units of an item.

From the above, (*a*) Sketch the graph of the function.

(*b*) Indicate the points of discontinuity.

(*c*) Find the limit of the function as $x \to 500$, $x \to 1000$ and $x \to 5000$. and

(*d*) Explain why the function is discontinuous at $x = 1000$ and $x = 5000$.

Ans. [(*b*) 1000 and 5,000, (*c*) 15,000, 30,000 and 26,000, 1,30,000 and 100,000 (*d*) L.H.L $\neq$ R.H.L]

2. A merchant displays the following price quotations for the different size of orders for his articles.

₹ 2.50 per kg for 20 kgs or less

₹ 2.00 per kg for more than 20 kgs. but not more than 50 kgs.

₹ 1.75 per kg for more than 50 kgs. but not more than 100 kgs.

₹ 1.50 per kg for more than 100 kgs.

Determine by the graphic method , if the price function of the merchant is continuous. Also, find the points of discontinuity, if any

Ans. [Discontinuous, $x = 20$, $x = 50$ and $x = 100$].

3. The cost function of a certain products is given by

$$C = \begin{bmatrix} 15x, & 0 \leq x \leq 2000 \\ 13x, & 2000 < x \leq 4000 \\ 10x, & x > 5000 \end{bmatrix}$$

(*a*) State by means of a graph, if the function is continuous.

(*b*) Point out the points of discontinuity if any.

Ans. [f(C) is discontinuous, at $x = 2000$ and $x = 4000$].

4. Show that $f(x) = x^2$ is continuous at $x = 2$.

5. Show that $f(x) = (x^2 + 3x + 4)$ is continuous at $x = 1$.

6. Prove that $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3 \\ 5, & \text{when } x = 3 \end{cases}$

is continuous at $x = 3$

7. Prove that $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{when } x \neq 5 \\ 10, & \text{when } x = 5 \end{cases}$

is continuous at $x = 5$.

8. Show that the function, $f(x)$ defined as $f(x) = \dfrac{(x+2)(x-1)}{(x-1)} = 3$, is continuous at $x = 1$.

9. Show that the function $x^2 + 4x - 2$ is continuous at $x = 1$.

10. Find, if the function $\dfrac{x^2 - 9}{x - 3}$ is discontinuous at $x = 3$.

11. Find, if the function $\dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$.

12. Examine the continuity of the following functions :

(i) $f(x) = \frac{1}{x}$ at $x = 0$ (ii) $f(x) = \frac{1}{x^2}$ at $x = 0$

(iii) $f(x) = \frac{x}{x}$ at $x \neq 0$.

13. Examine the continuity of the functions defined as follows at $x = 1$ and at $x = \frac{3}{2}$.

$$f(x) = \begin{cases} 3 + 2x, & \text{when } -\frac{3}{2} \leq x < 0 \\ 3 - 2x, & \text{when } 0 \leq x < \frac{3}{2} \\ -3 - 2x, & \text{when } x \geq \frac{3}{2} \end{cases}$$

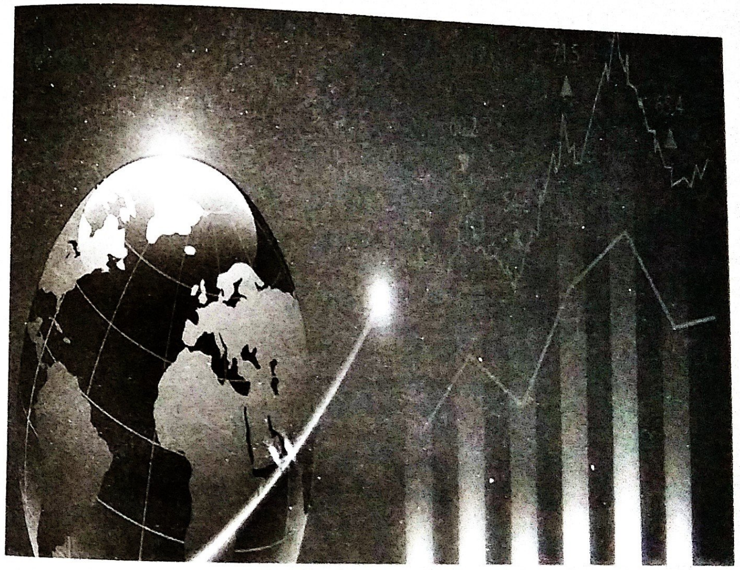
14. Prove that $f(x) = \begin{cases} 3 - x, & \text{when } x \leq 0 \\ x^2, & \text{when } x > 0 \end{cases}$

is discontinuous at $x = 0$.

□□□

Unit – II
(Derivative of a Function)

5. **Differentiation**
(With Application in Economics)



Chapter 5

DIFFERENTIATION

1. INTRODUCTION

The basic operations of calculus are differentiation and integration. The calculus is a study of mathematical analysis of change or movement. It has applications in all areas of Scientific interest. Calculus was developed in the 17th Century by **Sir Isac Newton (England) and Gottfried Leibnitz (Germany)** working independently. Newton's calculus originated from his attempts to solve some problems in physics and astronomy, where as Leibnitz's calculus originated from his attempt to solve some problems in geometry.

Differentiation is concerned with determining the rate of change of a given function. Integration is the inverse problem of finding the function when its rate of change is given. Calculus is an extremely valuable tool in solving problems relating to business and economics, as they are frequently concerned with change. Marginal analysis is one of the important applications of calculus in business and economics.

The rate at which a function changes with respect to independent variable is called the derivative of the function. The derivative of a function is defined in terms of the limit involving increments of the independent and dependent variables.

2. DIFFERENTIAL COEFFICIENT

By differential coefficient, we mean the derivative of a continuous function. This represents the limit of the ratio of increment in the dependent variable, say y corresponding to a small increment in the independent variable say, x as the latter tends to zero.

To make the point more clear, let $y = f(x)$. Let us suppose that when x is increased by a small increment Δx, then y increases by a small increment Δy. In such a case, we have,

$$y + \Delta y = f(x + \Delta x)$$

Thus, the change in the value of the function is,

$$(y + \Delta y) - y = f(x + \Delta_x) - f(x)$$

$$\Rightarrow \quad \Delta y = f(x + \Delta_x) - f(x)$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Where $\frac{\Delta y}{\Delta x}$ is the incremental ratio of the dependent variable y with respect to x, the independent variable. When $\Delta x \to 0$, and the limiting value of $\left[\frac{(f(x) + \Delta x) - f(x)}{\Delta x}\right]$, *i.e.*, $\frac{\Delta y}{\Delta x}$ exists, then we say that y

is differentiable with respect to x, and *the limiting value is called the differential coefficient, or the derivative of y with respect to x.*

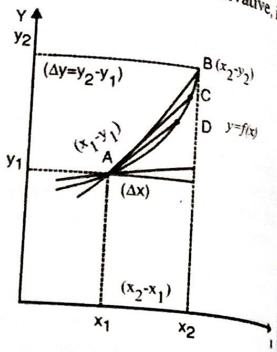
However, as a matter of convenience if is represented by

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

3. DERIVATIVE AND SLOPE OF A CURVE

The process of obtaining the derivative is called differentiation. When a function has a derivative, it is said to be differentiable.

The term derivative is a generalized expression for measuring the rate of change or slope of a function. Suppose, A and B are two points on the curve in a function $Y = f(x)$ whose Co-ordinates (x_1, y_1) and (x_2, y_2) respectively.



The given figure shows that, the average slope of the curve between two pointes A and B is measured by the slope of line joining the points A and B.

$$\text{Slope of the line AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta_y}{\Delta_x}.$$

The mathematical equation of the curve is given by

$$y = f(x)$$

then $\quad y_1 = \text{value } f(x) \text{ at } x = x_1$

$\quad = f(x_1)$

and $\quad y_2 = f(x_2)$

$$\therefore \quad \frac{\Delta_y}{\Delta_x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Here, $x_2, > x_1$ and thus $x_2 = x_1 + \Delta x_1$

Δx_1 is the small change of x_1.

$$f(x_2) = f(x_1 + \Delta x_1)$$

$$\frac{\Delta_y}{\Delta_x} = \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1}$$

This is the slope of a straight line AB, rather than the curve AB.

If we keep on making Δx_1, smaller we approach a point such as A and obtain a line that touches the curve only at the point A. The line is the tangent to the curve at the point A. Now Δx_1 is very small and the point B is extremely close to point A.

In mathematics, $\frac{\Delta_y}{\Delta_x}$ as $\Delta x \to 0$ is the slope of the curve at the point A.

$$\text{Slope at A} = \lim_{\Delta x \to 0} \left[\frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1}\right]$$

Thus, $\frac{dy}{dx}$ = rate of change in y w.r.t. x.

$$= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

The derivative of a function is the generalized expression for the slope of a function. At any point where the limit does not exist, the function $y = f(x)$ is said to have a derivative or to be differentiable and $\frac{dy}{dx}$ is said to be the first derivative or derivative of $y = f(x)$

Average rate of change and Instantaneous rate of change

The average rate of change of a function f over an interval x to $x + \Delta x$ is defined by the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limiting value of average rate of change of function $f(x)$ as $\Delta x \to 0$ is known as **instantaneous rate of change** of function $y = f(x)$ w.r.t. x.

The instantaneous rate of change of a function is known as derivative of the function.

If $y = f(x)$, then the derivative of y with respect to x is defined to be

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Provided this limit exists, if not the function is not differentiable.

4. DIFFERENTIATION FROM FIRST PRINCIPLE

If the derivative of a function is obtained by using the definition then it is called *differentiation from first principle.* They are illustrated as follows :

ILLUSTRATION 1. Differentiate the following functions by the method of first principle.

(i) x^2 (ii) $3x^2 + 2x$

(iii) $\frac{1}{x^2}$ (iv) $\sqrt{4 - x}$

SOLUTION

(*i*) We have $f(x) = x^2$. Thus, $f(x + h) = (x + h)^2$

By definition,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x + 0 = 2x$$

(*ii*) We have, $f(x) = 3x^2 + 2x$. Thus, $f(x + h) = 3(x + h)^2 + 2(x + h)$

By definition,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 + 2(x+h) - (3x^2 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 2x + 2h - 3x^2 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 3x^2 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \to 0} 6x + 3h + 2$$

$$= 6x + 3 \times 0 + 2 = 6x + 2$$

(*iii*) We have, $f(x) = \frac{1}{x^2}$ · Thus $f(x+h) = \frac{1}{(x+h)^2}$

By definition,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{(x+h)^2 . x^2 . h} = \lim_{h \to 0} \frac{(x - x - h)(x + x + h)}{(x+h)^2 . x^2 . h}$$

$$= \lim_{h \to 0} \frac{-h(2x+h)}{(x+h)^2 . x^2 . h} = \frac{\lim_{h \to 0} -(2x+h)}{\lim_{h \to 0} (x+h)^2 . x^2}$$

$$= \frac{-(2x+0)}{(x+0)^2 . x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3} = -2x^{-3}$$

(*iv*) We have, $f(x) = \sqrt{4-x}$. Thus, $f(x+h) = \sqrt{4-(x+h)} = \sqrt{4-x-h}$

By definition,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{4-x-h} - \sqrt{4-x}}{h} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0} \frac{(\sqrt{4-x-h} - \sqrt{4-x})(\sqrt{4-x-h} + \sqrt{4-x})}{h(\sqrt{4-x-h} + \sqrt{4-x})}$$

$$= \lim_{h \to 0} \frac{4 - x - h - 4 + x}{h(\sqrt{4-x-h} + \sqrt{4-x})} = \lim_{h \to 0} \frac{-h}{h(\sqrt{4-x-h} + \sqrt{4-x})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{4-x-h} + \sqrt{4-x}} = \frac{-1}{\sqrt{4-x-0} + \sqrt{4-x}} = -\frac{1}{2\sqrt{4-x}}$$

5. STANDARD FORMS

There are different standard functions (*viz.* Constant function, power function, polynomial functions, exponential function, logarithmic function, trigonometric function, inverse function, implicit function, composite function *etc.*) of a variable. For finding the differential coefficients of all such functions we are to apply the appropriate models that have been designed in the science of differentiation. These models are also known as STANDARD FORMS of differentiation.

They are presented along with illustrations as follows :

I. Differentiation of x^n, where $n \in \mathbf{R}$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Proof. We have, $f(x) = x^n$. Thus, $f(x+h) = (x+h)^n$

By the definition of derivatives (first principle), we get,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{z^n - x^n}{z - x}, \text{ where } z = x + h \text{ and } z \to x \text{ as } h \to 0$$

$$= nx^{n-1} \qquad \left[\text{using } \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Hence, $\frac{d}{dx}(x^n) = nx^{n-1}$

II. Differentiation of a constant function is zero

$$\boxed{\frac{d}{dx}(\mathrm{C}) = 0}$$

Proof. We have, $f(x) = \mathrm{C}$, Thus, $f(x+h) = \mathrm{C}$

[∵ constant is not dependent on any variable]

By the definition of derivatives we get,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\mathrm{C} - \mathrm{C}}{h} = \lim_{h \to 0} 0 = 0$$

Hence, $\frac{d}{dx}(\mathrm{C}) = 0$

III. Differentiation of a constant with any function

$$\boxed{\frac{d}{dx}[\mathrm{C}.f(x)] = \mathrm{C}.\frac{d}{dx}f(x)}$$

Proof. Let $g(x) = \mathrm{C}.f(x)$, then $g(x+h) = \mathrm{C}.f(x+h)$

By the definition we have,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\mathrm{C}.f(x+h) - \mathrm{C}.f(x)}{h}$$

$$= C \cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = C\cdot\frac{d}{dx}f(x)$$

Hence, $\frac{d}{dx}C.f(x) = C.\frac{d}{dx}f(x)$

ILLUSTRATION 2. Find the differential coefficients of the following Power functions :

(i) $y = x^5$, (ii) $y = \sqrt[3]{x}$, (iii) $y = \frac{1}{x^4}$,

(iv) $y = 200$, (v) $y = 2^3$. (vi) $y=18x^3$

SOLUTION

We have, $\frac{d}{dx}(x^n) = nx^{n-1}$, thus,

(*i*) when $y = x^5$

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$$

(*ii*) when $y = \sqrt[3]{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt[3]{x}) = \frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$$

(*iii*) when $y = \frac{1}{x^4}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}$$

(*iv*) when $y = 200$

$$\frac{dy}{dx} = \frac{d}{dx}(200) = 0$$

(*v*) when $y = 2^3$

$$\frac{dy}{dx} = \frac{d}{dx}(2^3) = 0$$

(vi) when $y = 18x^3$

$$\frac{dy}{dx} = \frac{d}{dx}(18x^3) = 18\frac{d}{dx}x^3$$

$$= 18 \times 3 \times x^{3-1} = 54x^2$$

IV. Differentiation of an exponential function e^x

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Proof. We have, $f(x) = e^x$. Thus, $f(x+h) = e^{x+h}$

By the definition we get,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x.e^h - e^x}{h} = e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) \quad [\because e^x \text{ does not contain } h]$$

$$= e^x . 1 \quad \left[\because \lim_{h\to 0}\frac{e^x-1}{x} = 1\right]$$

$$= e^x$$

Hence, $\frac{d}{dx}(e^x) = e^x$

Remark : $\frac{d}{dx}(e^{mx}) = m.e^{mx}$

V. Differentiation of a^x, where x and $a > 0$

$$\boxed{\frac{d}{dx}(a^x) = a^x \log_e a}$$

Proof. We have, $f(x) = a^x$. Thus, $f(x+h) = a^{x+h}$

By the definition we get,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x.a^h - a^x}{h}$$

$$= a^x . \lim_{h\to 0}\frac{a^h-1}{h} \quad (\because a^x \text{ does not contain } h)$$

$$= a^x . \log_e a \quad \left[\because \lim_{x\to 0}\frac{a^x-1}{x} = \log_e a\right]$$

Hence, $\frac{d}{dx}(a^x) = a^x . \log_e a$

Remark : $\frac{d}{dx}(a^{mx}) = m . a^{mx} . \log_e a$

VI. Differentiation of $\log_e x$, where $x > 0$

$$\boxed{\frac{d}{dx}(\log_e x) = \frac{1}{x}}$$

Proof. We have, $f(x) = \log_e x$. Thus, $f(x+h) = \log_e (x+h)$

By the definition we get,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\log_e(x+h)-\log_e x}{h}$$

$$= \lim_{h\to 0}\frac{\log_e\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{h/x}\times\frac{1}{x}$$

$$= 1\times\frac{1}{x} = \frac{1}{x}$$

$\left[\because \lim_{h\to 0}\frac{\log_e(1+x)}{x}\right.$

Hence, $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

VII. Differentiation of $\log_a x$ where $x > 0$, $a > 0$ $(a \neq 1)$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}}$$

Proof. We have, $f(x) = \log_a x$. thus, $f(x+h) = \log_a (x+h)$

By the definition we get,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$= \lim_{h\to 0}\frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\log_a\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\log_e\left(1+\frac{h}{x}\right)}{\log_e a.h}$$

$\left[\because \log_a z = \frac{\log_e z}{\log_e a}\right.$

$$= \frac{1}{\log_e a}\times\lim_{h\to 0}\frac{\log_e\left(1+\frac{h}{x}\right)}{h/x}\times\frac{1}{x}$$

$$= \frac{1}{\log_e a}\times 1\times\frac{1}{x} = \frac{1}{x\log_e a}$$

$\left[\because \log_a z = \frac{\log_e z}{\log_e a}\right.$

Hence, $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

ILLUSTRATION 3. Differentiate the following functions w.r.t.x

(i) e^{2x}, (ii) e^{7x}, (iii) 5^x, (iv) 10^x, (v) $\log 3\,x$, (vi) $\frac{1}{\log_x 5}$ (vii) 3^{2x}

SOLUTION

(i) $\frac{d}{dx}(e^{2x}) = e^{2x}.\frac{d}{dx}(2x) = e^{2x}.2 = 2e^{2x}$

(ii) $\frac{d}{dx}(e^{7x}) = e^{7x}.\frac{d}{dx}(7x) = e^{7x}\times 7 = 7e^{7x}$

(iii) $\frac{d}{dx}(5^x) = 5^x\log_e 5$ $\left[\because \frac{d}{dx}a^x = a^x\log_e a\right]$

(iv) $\frac{d}{dx}(10^x) = 10^x\log_e 10$ [∵ – do –]

(v) $\frac{d}{dx}(\log 3^x) = \frac{1}{x\log_e 3}$ $\left[\because \frac{d}{dx}\log_a x = \frac{1}{x\log_e a}\right]$

(vi) $\frac{d}{dx}\left(\frac{1}{\log_x 5}\right) = \frac{d}{dx}(\log_5^x) = \frac{1}{x\log_e 5}$ [∵ – do–]

(vii) $\frac{d}{dx}3^{2x} = 2\times 3^{2x}\times\log_e 3 = 2.3^{2x}.\log_e 3$

6. THEOREMS ON DIFFERENTIATION

Theorem – 1 (Summation rule)

The derivative of the sum of two differentiable functions is the sum of their derivative, i.e., if u, v be two differentiable functions of x, then

$$\boxed{\frac{d}{dx}(u\pm v) = \frac{du}{dx}\pm\frac{dv}{dx}}$$

Proof. Let $u = m(x)$, $v = n(x)$ and $f(x) = u + v$

Then, $f(x) = m(x) + n(x)$. Thus $f(x+h) = m(x+h) + n(x+h)$

By the definition of derivatives

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{m(x+h)+n(x+h)-[m(x)+n(x)]}{h}$$

$$= \lim_{h\to 0}\frac{m(x+h)+n(x+h)-m(x)-n(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{m(x+h)-m(x)}{h}+\frac{n(x+h)-n(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{m(x+h)-m(x)}{h} + \lim_{h\to 0}\frac{n(x+h)-n(x)}{h}$$

$$= \frac{du}{dx}+\frac{dv}{dx}$$

Hence, $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$

Similarly, if $f(x)=u-v$, then

$$\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$$

Note. The above formulae can be extended to a sum of three or more differentiable functions. Hence the derivative of a sum of several differentiable functions is the algebraic sum of the derivatives of the individual functions.

i.e., if $y=u\pm v\pm w\pm z\pm \ldots$ then

$$\frac{dy}{dx}=\frac{du}{dx}\pm\frac{dv}{dx}\pm\frac{dw}{dx}\pm\frac{dz}{dx}\pm\ldots$$

ILLUSTRATION 4. Find the differential coefficients of the following functions w.r.t. x.

(i) $y=5x+x^3$ (ii) $y=\frac{(1-x)^2}{x^2}$

(iii) $y=\frac{1}{x}+\sqrt{x}+e^x$ (iv) $y=x^5+2x-x^3$

(v) $y=x^4+2e^x+3$ (vi) $y=2x^{3/2}-3\log_e^x+3x^2$

SOLUTION

(*i*) When $y=5x+x^3$,

$$\frac{dy}{dx}=\frac{d}{dx}(5x+x^3)=\frac{d}{dx}(5x)+\frac{d}{dx}(x^3)=5\frac{d}{dx}(x)+\frac{d}{dx}x^3$$

$$=(5\times 1\times x^{1-1})+(3\times x^{3-1})=5+3x^2$$

(*ii*) When $y=\frac{(1-x)^2}{x^2}$

$\Rightarrow$ $y=\frac{1-2x+x^2}{x^2}=\frac{1}{x^2}-\frac{2}{x}+1=x^{-2}-2x^{-1}+1$

Thus, $\frac{dy}{dx}=\frac{d}{dx}(x^{-2}-2x^{-1}+1)$

$$=\frac{d}{dx}(x^{-2})-2.\frac{d}{dx}x^{-1}+\frac{d}{dx}(1)$$

$$=-2.x^{-2-1}-2(-1).x^{-1-1}+0$$

$$=-2x^{-3}+2x^{-2}=-2x^{-2}(x^{-1}-1)$$

(*iii*) When $y=\frac{1}{x}+\sqrt{x}+e^x$,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{1}{x}+\sqrt{x}+e^x\right]=\frac{d}{dx}(x^{-1})+\frac{d}{dx}x^{1/2}+\frac{d}{dx}e^x$$

$$=-x^{-2}+\frac{1}{2}x^{-1/2}+e^x$$

$$=-\frac{1}{x^2}+\frac{1}{2\sqrt{x}}+e^x$$

(*iv*) When, $y=x^5+2x-x^3$

$$\frac{dy}{dx}=\frac{d}{dx}(x^5)+2.\frac{d}{dx}(x)+\frac{d}{dx}(x^3)$$

$$=5.x^{5-1}+2.x^{1-1}+3.x^{3-1}$$

$$=5x^4+2+3x^2$$

(*v*) When $y=x^4+2e^x+3$

$$\frac{dy}{dx}=\frac{d}{dx}(x^4+2e^x+3)$$

$$=\frac{d}{dx}(x^4)+2.\frac{d}{dx}(e^x)+\frac{d}{dx}(3)$$

$$=4.x^{4-1}+2.e^x+0=4x^3+2e^x$$

$$=2(2x^3+e^x)$$

(*vi*) where, $y=2x^{3/2}-3\log_e e^x+6$

$$\frac{dy}{dx}=\frac{d}{dx}(2x^{3/2}-3\log_e x+6)=2.\frac{d}{dx}x^{3/2}-3.\frac{d}{dx}\log_e x+\frac{d}{dx}(6)$$

$$=2\times\frac{3}{2}\times x^{\frac{3}{2}-1}-3\times\frac{1}{x}+0=3x^{1/2}-3.\frac{1}{x}=3\left(\sqrt{x}-\frac{1}{x}\right)$$

EXERCISE (A)

1. Differentiate the following functions *w.r.t.* x from the first principle :

(*i*) x (*ii*) $3x+2$ (*iii*) x^2-2x

(*iv*) e^{2x} (*v*) 2^x (*vi*) $\frac{1}{\sqrt{x}}$ $(x>0)$

(*vii*) $2x^2+3x$ (*viii*) x^3 (*ix*) $\sqrt{x^2+1}$

Ans. [(*i*) 1, (*ii*) 3, (*iii*) $2x-2$, (*iv*) $2e^{2x}$, (*v*) $2^x\log_e 2$, (*vi*) $-\frac{1}{2x^{3/2}}$, (*vii*) $4x+3$, (*viii*) $3x^2$, (*ix*) $\frac{x}{\sqrt{x^2+1}}$]

2. Find the differential coefficients of the following functions *w.r.t.* x (Use Standard Form) :

(*i*) x^7 (*ii*) $\frac{1}{x^3}$ (*iii*) $\sqrt{x}$ (*iv*) $\frac{1}{\sqrt{x}}$ (*v*) $\log_5 x$

(*vi*) e^{-x} (*vii*) e^{3x} (*viii*) e^{ax+b} (*ix*) $\frac{1}{x^{3/2}}$ (*x*) $\frac{1}{\log_x 7}$

Ans. $\left[\right.$ (i) $7x^6$, (ii) $-3x^{-4}$, (iii) $\frac{1}{2\sqrt{x}}$, (iv) $-\frac{1}{2x^{3/2}}$, (v) $\frac{1}{x \log_e 5}$, (vi) $-e^{-x}$, (vii) $3e^{3x}$,

(viii) ae^{ax+b}, (ix) $-\frac{3}{2x^{5/2}}$, (x) $\frac{1}{x \log_e 7}$. $\left. \right]$

3. Find the differential coefficients of the following functions with respect to x :

(i) $x + \frac{1}{x}$ (ii) $(x-2)^2$ (iii) $(2x+3)^2$

(iv) $x^5 + x^3 + \frac{1}{x}$ (v) $x^{11} - x^{10} + 5$ (vi) $2x^3 + 3x^2 - 6x + 4$

(vii) $2\sqrt{x} - \frac{3}{x} + \frac{4}{x^2} - \frac{1}{x^3}$ (viii) $(x^2 - 2x)^2$ (ix) $\frac{(x-1)^3}{x^2}$

(x) $3x^{-2/3} + 4x^{-1/2} - 2$ (xi) $3x^{2/3} - 4x^{3/4} + 5x^{4/5}$ (xii) $x^4 + 4^x + \log x$

(xiii) $2x^{3/2} - 3e^{3x} + 2^x$ (xiv) $3 \log x - 2\sqrt{x} + 5^x - 2$ (xv) $3^{4x} + \frac{3}{\sqrt[3]{x}}$

Ans. $\left[\right.$ (i) $\left(1 - \frac{1}{x^2}\right)$, (ii) $(2x-2)$, (iii) $8x + 12$, (iv) $5x^4 + 3x^2 - \frac{1}{x^2}$, (v) $11x^{10} - 10x^9$, (vi) $6(x^2 +$ $- 1)$, (vii) $\frac{1}{\sqrt{x}} + \frac{3}{x^2} - \frac{8}{x^3} + \frac{3}{x^4}$, (viii) $4x(x-1)(x-2)$, (ix) $1 - \frac{3}{x^2} + \frac{2}{x^3}$

(x) $-2(x^{-5/3} + x^{-3/2})$, (xi) $2x^{-1/3} - 3x^{-1/4} + 4x^{-1/5}$, (xii) $4x^3 + 4^x \log_e 4 + \frac{1}{x}$,

(xiii) $3\sqrt{x} - e^{3x} + 2^x \log_e$, (xiv) $\frac{3}{x} - \frac{1}{\sqrt{x}} + 5^x \log_e 5$, (xv) $4 \cdot 3^{4x} \cdot \log_e 3 - x^{\frac{-4}{3}}$ $\left. \right]$

Theorem – 2 (Product rule)

The derivative of the product of the two functions is equal to the product of the second function an the derivative of the first function plus the product of the first function and the derivative of the second function.

i.e., if, u and v be two differentiable functions of x, then

$$\boxed{\frac{d}{dx}(uv) = v\frac{d}{dx}(u) + u\frac{d}{dx}(v)}$$

Proof. Let $f(x) = uv$, where $u = m(x)$ and $v = n(x)$. Then $f(x) = m(x) \cdot n(x)$. Thus, $f(x+h) = m(x+h) \cdot n(x+h)$.

By the definition we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{m(x+h).n(x+h) - m(x).n(x)}{h}$$

$$= \lim_{h \to 0} \frac{m(x+h).n(x+h) - m(x+h).n(x) + m(x+h).n(x) - m(x).n(x)}{h}$$

$$= \lim_{h \to 0} \left[m(x+h) \cdot \frac{n(x+h) - n(x)}{h} \right] + \left[n(x) \frac{m(x+h) - m(x)}{h} \right]$$

$$= \left[\lim_{h \to 0} m(x+h) . \lim_{h \to 0} \frac{n(x+h) - n(x)}{h} \right] + \left[n(x) \lim_{h \to 0} \frac{m(x+h) - m(x)}{h} \right]$$

$$= m(x) \times \frac{dv}{dx} + n(x) \cdot \frac{du}{dx}$$

$$= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} = v \cdot \frac{du}{dx} + u \cdot \frac{dv}{dx}$$

Hence, $\frac{d}{dx}(uv) = v.\frac{du}{dx} + u.\frac{dv}{dx}$

Note. The above formula can be extended to the product of more than two functions. For example if $y = uvw$, a product of three differentiable functions of x, then

$$\frac{d}{dx}(uvw) = vw\frac{d}{dx}(u) + u\frac{d}{dx}(vw)$$

$$= vw\frac{d}{dx}(u) + u\left[w\frac{d(v)}{dx} + v\frac{d(w)}{dx} \right]$$

$$= vw\frac{d}{dx}(u) + uw\frac{d}{dx}(v) + uv\frac{d}{dx}(w)$$

ILLUSTRATION 5. Find the differential coefficients of the following functions w.r.t. x.

(i) $x^3 e^x$ (ii) $5^x . x^5$ (iii) $x^2 . \log x$ (iv) $e^x \log x$

(v) $x^5 . e^{2x} . \log x$ (vi) $e^x \log x (2x^2 + 3)$ (vii) $\log_a x$ (viii) $(x^2 + 1)(3x^2 - 2x^3)$

(ix) $(2x^3 + 3)(3x^2 + 2x + 9)$

SOLUTION

(i) Let $y = x^3 . e^x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(x^3 . e^x) = e^x . \frac{d}{dx}(x^3) + x^3 . \frac{d}{dx}(e^x)$$

$$= e^x . 3x^2 + x^3 e^x = x^2 e^x (3 + x)$$

(ii) Let $y = 5^x . x^5$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(5^x \cdot x^5) = x^5 . \frac{d}{dx} 5^x + 5^x \cdot \frac{d}{dx} x^5$$

$$= x^5 . 5^x . \log_e 5 + 5^x . 5 x^{5-1} = 5^x x^5 . \log_e 5 + 5^x . 5 x^4$$

$$= 5^x x^4 (x \log_e 5 + 5)$$

(iii) Let $y = x^2 \log_e x$

$$\therefore \quad \frac{d}{dx} = \frac{d}{dx}(x^2 \cdot \log_e x) = \log_e x . \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}(\log_e x)$$

$$= \log_e x . 2x^{2-1} + x^2 . \frac{1}{x} = 2x \log_e x + x$$

$$= x(2 \log_e x + 1)$$

(iv) Let $y = e^x \log x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^x.\log x) = \log x.\frac{d}{dx}(e^x) + e^x.\frac{d}{dx}(\log x)$$

$$= \log x.e^x + e^x.\frac{1}{x}$$

$$= e^x\left(\log + \frac{1}{x}\right)$$

(v) $y = x^5 .e^{2x}.\log x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(x^5 \cdot e^{2x}.\log x) = (e^{2x}.\log x)\cdot\frac{d}{dx}(x^5) + x^5\cdot\frac{d}{dx}(e^{2x}.\log x)$$

$$= e^{2x}.\log x.5.x^{5-1} + x^5[\log x.\frac{d}{dx}(e^{2x}) + e^{2x}.\frac{d}{dx}(\log x)]$$

$$= e^{2x}.\log x.5x^4 + x^5[\log x.2.e^{2x} + e^{2x}.\frac{1}{x})$$

$$= e^{2x}.\log x.5x^4 + 2e^{2x}x^5\log x + e^{2x}.\frac{1}{x}\times x^5)$$

$$= x^4 e^{2x}(5\log x + 2x\log x + 1)$$

(vi) Let $y = e^x.\log x.(2x^2+3)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[e^x.\log x.(2x^2+3)] = \log x.(2x^2+3).\frac{d}{dx}(e^x) + e^x\frac{d}{dx}\log x.(2x^2+3)$$

$$= \log x.(2x^2+3).e^x + e^x[(2x^2+3)\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(2x^2+3)$$

$$= \log x.(2x^2+3).e^x + e^x[(2x^2+3)].\frac{1}{x} + \log x(4x+0)$$

$$= \log x(2x^2+3).e^x + e^x.(2x^2+3).\frac{1}{x} + 4x.\log x$$

$$= e^x\left[2x^2\log x + 3\log x + 2x + \frac{3}{x} + 4x.\log x\right]$$

(vii) Let $y = \log_a x$

$$\therefore \quad y = \frac{\log_e x}{\log_e a} \qquad \left[\because \log_a x = \frac{\log_e x}{\log_e a}\right]$$

Thus,

$$\frac{dy}{dx} = \left(\frac{\log_e x}{\log_e a}\right) = \frac{d}{dx}\left(\log_e x \times \frac{1}{\log_e a}\right) \qquad \left[(\log_e x)\left(\frac{1}{\log_e a}\right)\right]$$

$$= \frac{1}{\log_e a}\cdot\frac{d}{dx}(\log_e x) = \frac{1}{\log_e a}\times\frac{1}{x} = \frac{1}{x\log_e a} \qquad \left[\because \frac{1}{\log_e a} \text{ is a constant and } \frac{d}{dx}(C) = 0\right]$$

(viii) Let $y = (x^2+1)(3x^2-2x^3)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(x^2+1)(3x^2-2x^3)]$$

$$= (3x^2-2x^3)\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}(3x^2-2x^3)$$

$$= (3x^2-2x^3).2x + (x^2+1).(6x-6x^2)$$

$$= 6x^3 - 4x^4 + 6x^3 + 6x - 6x^4 - 6x^2$$

$$= -10x^4 + 12x^3 - 6x^2 + 6x$$

$$= -2x(5x^3 - 6x^2 + 3x - 3)$$

(ix) Let $y = (2x^3+3)(3x^2+2x+9)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(2x^3+3)(3x^2+2x+9)]$$

$$= (3x^2+2x+9)\frac{d}{dx}(2x^3+3) + (2x^3+3)\frac{d}{dx}(3x^2+2x+9)$$

$$= (3x^2+2x+9)(6x^2) + (2x^3+3)(6x+2)$$

$$= 18x^4 + 12x^3 + 54x^2 + 12x^4 + 18x + 4x^3 + 6$$

$$= 30x^4 + 16x^3 + 54x^2 + 18x + 6 = 2(15x^4 + 8x^3 + 27x^2 + 9x + 3)$$

Theorem – 3 (Quotient rule)

The derivations of the quotient of any two functions is equal to the product of the denominator and the derivative of the numerator minus the product of the numerator and the derivative of the denominator, all divided by the square of the denominator.

i.e., if, u and v be two differentiable functions of x, and v ≠ 0, then

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}}$$

Proof. Let $f(x) = \frac{u}{v}$, where $mx) = u$ and $n(x) = v$

Then, $f(x) = \frac{m(x)}{n(x)}$. Thus, $f(x+h) = \frac{m(x+h)}{n(x+h)}$

By the definition we have,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{m(x+h)}{n(x+h)} - \frac{m(x)}{n(x)}}{h}$$

$$= \lim_{h\to 0}\frac{m(x+h).n(x) - m(x).n(x+h)}{n(x+h).n(x).h}$$

$$= \lim_{h\to 0} \frac{m(x+h).n(x) - m(x).n(x) + m(x).n(x) - m(x).n(x+h)}{n(x+h).n(x).h}$$

$$= \lim_{h\to 0} \frac{1}{n(x+h).n(x)}\left[n(x)\,\frac{m(x+h)-m(x)}{h} - m(x)\,.\,\frac{n(x+h)-n(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{1}{n(x+h)n(x)}\left[n(x)\lim_{h\to 0}\frac{m(x+h)-m(x)}{h} - m(x)\lim_{h\to 0}\frac{n(x+h)-n(x)}{h}\right]$$

$$= \frac{1}{[n(x)]^2}\left[n(x)\,\frac{du}{dx} - m(x)\,\frac{dv}{dx}\right]$$

$$= \frac{1}{v^2}\left[v\,.\,\frac{du}{dx} - u\,.\,\frac{dv}{dx}\right]$$

$$= \frac{v.\frac{d}{dx}(u) - u.\frac{dv}{dx}}{v^2}$$

Hence, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{d}{dx}(u) - u.\frac{dv}{dx}}{v^2}$

ILLUSTRATION 6. Determine the differential coefficients of the quotient of the following functions w.r.t. x.

(i) $y = \dfrac{x^3}{x^3+2}$ (ii) $y = \dfrac{5}{1-3x}$ (iii) $y = \dfrac{3x+4}{e^x+2}$ (iv) $y = \dfrac{\sqrt{x}+2}{\sqrt{x}}$

SOLUTION

(i) If $y = \dfrac{x^3}{x^3+2}$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3}{x^3+2}\right) = \frac{(x^3+2)\frac{d}{dx}(x^3) - x^3.\frac{d}{dx}(x^3+2)}{(x^3+2)^2}$$

$$= \frac{(x^3+2).3x^2 - x^3.(3x^2)}{(x^3+2)^2} = \frac{3x^5+6x^2-3x^5}{(x^3+2)^2} = \frac{6x^2}{(x^3+2)^2}$$

(ii) If $y = \dfrac{5}{1-3x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{5}{1-3x}\right) = \frac{(1-3x)\frac{d}{dx}(5) - 5.\frac{d}{dx}(1-3x)}{(1-3x)^2}$$

$$= \frac{(1-3x)\times 0 - (5\times -3)}{(1-3x)^2} = \frac{15}{(1-3x)^2}$$

(iii) If $y = \dfrac{3x+4}{e^x+2}$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x+4}{e^x+2}\right) = \frac{(e^x+2).\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(e^x+2)}{(e^x+2)^2}$$

$$= \frac{(e^x+2).(3) - (3x+4).e^x}{(e^x+2)^2}$$

$$= \frac{3e^x+6-3xe^x-4e^x}{(e^x+2)^2} = \frac{-e^x(-3+3x+4)+6}{(e^x+2)^2}$$

$$= \frac{-e^x(1+3x)+6}{(e^x+2)^2}$$

(iv) If $y = \dfrac{\sqrt{x}+2}{\sqrt{x}}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sqrt{x}+2}{\sqrt{x}}\right) = \frac{\sqrt{x}\frac{d}{dx}(\sqrt{x}+2) - (\sqrt{x}+2)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}.\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) - (\sqrt{x}+2)\frac{1}{2}x^{\frac{1}{2}-1}}{x} = \frac{\left(\sqrt{x}.\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}+2).\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}-\sqrt{x}-2)}{x} = \frac{-1}{\sqrt{x}.x} = \frac{-1}{x^{3/2}} = -x^{-3/2}$$

EXERCISE (B)

1. Find the derivatives of each of the following functions :

(i) $x\,.\,e^x$ (ii) $x^2\,.\,\log x$ (iii) $4^x\,.\,x^4$ (iv) $10^x\,.\,x^{16}$

(v) $\sqrt{x}\,.\,e^{3x}$ (vi) $x^3\,.\,e^{2x}\,.\,\log x$ (vii) $(2x-3)(4x-5)$ (viii) $(x^2-2)(x^3+7)$

Ans. [(i) $e^x(1+x)$, (ii) $x(1+2\log_e x)$, (iii) $4^x x^3(4+x\log_e 4)$, (iv) $10^x.x^{15}(x\log_e^{10}+16)$,

(v) $e^{3x}\left(\dfrac{1}{2\sqrt{x}}+3\sqrt{x}\right)$, (vi) $x^2e^{2x}(3\log x+2x\log x+1)$, (vii) $2(8x-11)$, (viii) $5x^4-6x^2+14x$]

2. Differentiate each of the following functions with respect to x :

(i) $\dfrac{x^2}{e^x}$ (ii) $\dfrac{\log x}{x^3}$ (iii) $\dfrac{3}{1-5x}$ (iv) $\dfrac{x^2+1}{x^2-1}$

(v) $\dfrac{5-4x}{5+4x}$ (vi) $\dfrac{x^2-1}{x^2+7x+1}$

Ans. $\left[(i)\ \frac{x(2-x)}{e^x},\ (ii)\ \frac{1-3\log x}{x^4},\ (iii)\ \frac{15}{(1-5x)^2},\ (iv)\ \frac{-4x}{(x^2-1)^2},\ (v)\ \frac{-40}{(5+4x)^2},\right.$

$\left.(vi)\ \frac{7x^2+4x+7}{(x^2+7x+1)^2}\right]$

7. DIFFERENTIATION OF FUNCTION OF A FUNCTION (CHAIN-RULE)

If y is a function of u, where u itself is a function of x, then y is called the function of a function, or a composite function of x.

The derivative of the composite function y with respect to x is equal to the product of the derivative of y with respect to u and derivative of u with respect to x. Thus, if $y = f(u)$ and $u = f(x)$, the model will appear as under :

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

The above model can be extended to the derivative of a composite function of long chain. Thus, if $y = f(u)$, $u = f(v)$, and $v = f(x)$, the model will be modified as under :

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}}$$

ILLUSTRATION 7. Find the differential coefficients of the following composite functions :

(i) $(3x+2)^4$ (ii) $(3+2x^2)^5$ (iii) 7^{x^2+2x}

(iv) $\log(3x+7)$ (v) e^{x^2+3x+9} (vi) $\frac{1}{\sqrt{5x^3-9x^2+7}}$

SOLUTION

(i) Let $y = (3x+2)^4$.

If, $u = (3x+2)$, then $y = u^4$

Now, $\frac{du}{dx} = \frac{d}{dx}(3x+2) = 3+0 = 3$

And $\frac{dy}{du} = \frac{d}{du}(u^4) = 4u^3 = \frac{d}{du}u^4 = 4u^4 = 4(3x+2)^3$

By the model of a composite function, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 4(3x+2)^3 \times 3 = 12(3x+2)^3$$

(ii) Let $y = (3+2x^2)^5$

If, $(3+2x^2) = u$, then $y = u^5$

Now, $\frac{du}{dx} = \frac{d}{dx}(3+2x^2) = 0+4x = 4x$

And $\frac{dy}{du} = \frac{d}{du}(4^5) = 5u^4 = 5(3+2x^2)^4$

By the chain rule we have, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 5(3+2x^2)^4 \times 4x = 20x(3+2x^2)^4$$

(iii) Let $y = 7^{x^2+2x}$

If $u = x^2+2x$, then $y = 7^u$

Now $\frac{du}{dx} = \frac{d}{dx}(x^2+2x) = \frac{d}{dx}(x^2) + 2.\frac{d}{x}(x) = 2x+2 = 2(x+1)$

$\frac{dy}{du} = \frac{d}{du}(7^u) = 7^u.\log_e 7 = 7^{x^2+2x}.\log_e 7$

By chain rule we have,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 7^{x^2+2x}.\log_e 7.2(x+1)$$

(iv) Let $y = \log(3x+7)$

If $u = 3x+7$, then $y = \log u$

Now, $\frac{du}{dx} = \frac{d}{dx}(3x+7) = 3+0 = 3$

$\frac{dy}{du} = \frac{d}{dx}(\log u) = \frac{1}{u} = \frac{1}{3x+7}$

By Chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{3x+7} \times 3 = \frac{3}{3x+7}$$

(v) Let $y = e^{x^2+3x+9}$

If $x^2+3x+9 = u$, then $y = e^u$

Now, $\frac{du}{dx} = \frac{d}{dx}(x^2+3x+9) = 2x+3$

$\frac{dy}{du} = \frac{d}{du}(e^u) = e^u = e^{x^2+3x+9}$

By chain rule we have,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^{x^2+3x+9}.(2x+3)$$

(vi) Let $y = \frac{1}{\sqrt{5x^3-9x^2+7}}$

Let $u = 5x^3-9x^2+7$

Then $$y=\frac{1}{\sqrt{u}}=u^{-1/2}$$

$$\frac{du}{dx}=\frac{d}{dx}(5x^3-9x^2+7)=15x^2-18x$$

$$\frac{dy}{du}=\frac{d}{du}(u^{-1/2})=\frac{-1}{2}u^{\frac{-1}{2}-1}=\frac{-1}{2}u^{-3/2}=-\frac{1}{2}(5x^3-9x^2+7)^{-3/2}$$

By chain rule we have $$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-\frac{1}{2}(5x^3-9x^2+7)^{-3/2}.(15x^2-18x)$$

$$=\frac{18x-15x^2}{2(5x^3-9x^2+7)^{3/2}}$$

ILLUSTRATION 8. Find the differential coefficient of the following functions :

(i) $y=(x^3+3)e^{5x}$ (ii) $y=\sqrt{2x}+3^{2x}$

(iii) $x=\sqrt{(y^2+1)}$, (iv) $y=\log\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$

SOLUTION

(i) When $y=(x^3+3)e^{5x}$

$$\frac{dy}{dx}=\frac{d}{dx}\left[(x^3+3).e^{5x}\right]$$

$$=e^{5x}.\frac{d}{dx}(x^3+3)+(x^3+3)\frac{d}{dx}(e^{5x})$$

$$=e^{5x}(3x^2)+(x^3+3).5.e^{5x}$$

$$=e^{5x}(3x^2+5x^3+15)=e^{5x}(5x^3+3x^2+15)$$

(ii) When $y=\sqrt{2x}+3^{2x}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\sqrt{2x}+3^{2x})=\frac{d}{dx}\left(\sqrt{2}.\sqrt{x}\right)+\frac{d}{dx}(3^{2x})$$

$$=\sqrt{2}.\frac{1}{2}.x^{-1/2}+2.3^{2x}.\log_e 3=\frac{1}{\sqrt{2x}}+2.3^{2x}.\log_e 3$$

(iii) When $x=\sqrt{y^2+1}$,

$$\Rightarrow \quad x^2=y^2+1 \quad \text{or} \quad y=\sqrt{x^2-1}$$

Let $u=x^2-1$, Thus, $y=\sqrt{u}$

$$\frac{du}{dx}=\frac{d}{dx}(x^2-1)=2x$$

$$\frac{dy}{du}=\frac{d}{du}(\sqrt{u})=\frac{1}{2}u^{-1/2}=\frac{1}{2}(x^2-1)^{-1/2}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}(x-1)^{-1/2}\times 2x$$

$$=x(x^2+1)^{-1/2}=\frac{x}{\sqrt{x^2+1}}$$

(iv) When $y=\log\left[e^x.\left(\frac{x-2}{x+2}\right)^{3/4}\right]$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\log\left[e^x.\left(\frac{x-2}{x+2}\right)^{3/4}\right]=\frac{d}{dx}\left[\log e^x+\log\left(\frac{x-2}{x+2}\right)^{3/4}\right]$$

$$=\frac{d}{dx}\left[x\log e+\frac{3}{4}\{\log(x-2)-\log(x+2)\}\right]$$

$$=\frac{d}{dx}\left[x+\frac{3}{4}(\log(x-2)-\log(x+2))\right] \quad [\because \log_e e=1]$$

$$=\frac{d}{dx}(x)+\frac{3}{4}\left[\frac{d}{dx}\log(x-2)-\frac{d}{dx}\log(x+2)\right]$$

$$=1+\frac{3}{4}\left[\frac{1}{x-2}-\frac{1}{x+2}\right]=1+\frac{3}{4}\left(\frac{x+2-x+2}{(x-2)(x+2)}\right)$$

$$=1+\left(\frac{3}{4}\times\frac{4}{x^2-4}\right)=\frac{x^2-4+3}{x^2-4}=\frac{x^2-1}{x^2-4}$$

EXERCISE (C)

1. Differentiate the following, with respect to x :

(i) $(5-2x)^4$ (ii) $\frac{1}{(5-2x)^3}$ (iii) $\sqrt{5-2x}$ (iv) $\frac{1}{\sqrt{5-2x}}$

(v) $(3x^2-8x+9)^2$ (vi) $\frac{1}{(3x^2-8x+9)}$ (vii) $\left(x+\frac{1}{x}\right)^3$ (viii) $\frac{1}{(7-3x+x^3)^3}$

(ix) 2^{x^3} (x) 3^{e^x} (xi) $3^{x\log x}$ (xii) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$

(xiii) $\sqrt{\frac{1-x^2}{1+x^2}}$ (xiv) 3^{x^2+2x}

Ans. [(i) $-8(5-2x)^3$ (ii) $\frac{6}{(5-2x)^4}$ (iii) $-\frac{1}{\sqrt{5-2x}}$

(iv) $\frac{1}{(5-2x)^{3/2}}$ (v) $4(3x^2-8x+9)(3x-4)$ (vi) $-\frac{2(3x-4)}{(3x^2-8x+9)^2}$

(vii) $3\left(3+\frac{1}{x}\right)^2\left(1+\frac{1}{x^2}\right)$ (viii) $\frac{-3(-3+3x^2)}{(7-3x+x^3)}$ (ix) $3x^2.2^{x^3}\log 2$

(x) $3^{e^x}\log 3 \,.\, e^x$ (xi) $3^{x\log x} \,.\, \log 3\,(\log x + 1)$ (xii) $\dfrac{-2a^2x}{\sqrt{a^2-x^2}\,(a^2+x^2)^{3/2}}$

(xiii) $\dfrac{-2x}{\sqrt{1-x^2}\,(1+x^2)^{3/2}}$ (xiv) $(3x^2 + 2x \,.\, \log 3)\,.\,(2x+2)$]

2. Find the different coefficient of the following functions :

(i) $(x^2+5)^{3/2}$ (ii) $(5+2x-3x^2)^{1/2}\log x$

(iii) $\log\,[5-2x)\,(5+3x)]$ (iv) $\log\left(\dfrac{6+7x}{7+8x^2}\right)$

(v) $\log\sqrt[3]{5-3x^2}$ (vi) $e^{5x}(x^2+5)^{-3\backslash 2}$

(vii) x^2e^{-2x} (viii) $\log\sqrt{x+(x^2+1)}$

(ix) $\dfrac{e^x\log x}{x^2}$ (x) $\dfrac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$

(xi) $\log\left(\dfrac{x^2+x+1}{x^2-x+1}\right)$ (xii) $\log\,(3x+2) - x^2\log\,(2x-1)$

Ans. (i) $3x(x^2+5)^{1/2}$ (ii) $\dfrac{(5+2x-3x^2)+(x-3x^2)\log x}{x(5+2x-3x^2)^{1/2}}$ (iii) $\dfrac{5-12x}{(5-2x)\,(5+3x)}$

(iv) $\dfrac{49-96x-56x^2}{(6+7x)\,(7+8x^2)}$ (v) $\dfrac{-2x}{5-3x^2}$ (vi) $\dfrac{e^{5x}(5x^2-3x+25}{(x^2+5)^{5/2}}$

(vii) $2xe^{-2x}\,(1-x)$ (viii) $\dfrac{1}{\sqrt{x^2+1}}$ (ix) $e^x.x^{-2}\left(\log x+\dfrac{1}{x}-\dfrac{2}{x}\log x\right)$

(x) $\dfrac{-8}{(e^{2x}-e^{-2x})^2}$ (xi) $-\dfrac{2(x^2-1)}{x^4+x^2+1}$ (xii) $\dfrac{3}{3x+2}-\dfrac{2x^2}{2x-1}-2x\log\,(2x-1)$

3. Find the differential coefficient of y with respect to x, when

(i) $y = (x^2+1)\,(3x^2-2x)^3$ (ii) $y = \log_a(x^2+1)$.

Ans. $\left[(i)\ -2x\,(5x^3-6x^2+3x-3),\ (ii)\ \dfrac{2x}{(x^2+1)\log_e a}\right]$

8. LOGARITHMIC DIFFERENTIATION

If the power (or exponent) of a function of x is also, a function of x, or if the function is a m function, or a product of a number of functions, then it would be convenient for us to take logarithms respect to the base 'e' and then differentiate both sides $w.r.t.$ x.

Remark : **(1)** $\dfrac{dy}{dx}(\log y) = \dfrac{1}{y}\cdot\dfrac{dy}{dx}$

(2) $u^v = e^{v\log u}$

let $z = u^v$, then $\log z = v\log u$

$\therefore\ z = e^{v\log u}$, $i.e.$ $u^v = e^{v\log u}$

The following illustrations will make the idea clear.

ILLUSTRATION 9. Differentiate the following functions w.r.t. x.

(i) x^x (ii) x^{e^x} (iii) $x^{\sqrt{x}}$ (iv) x^{x^x} (v) $(x^x)^x$

SOLUTION

(i) Let $y = x^x$.

Then, taking logarithms of both the sides we get,

$$\log y = x\log x$$

Differentiating both the sides $w.r.t.$ x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log x)$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log x\,.\,\frac{d}{dx}(x) + x\,.\,\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = (\log x\times 1) + \left(x.\frac{1}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\,(\log x + 1) = x^x\,(1+\log x)$$

Aliter : Let $\quad y = x^x = e^{x\log x}$ $\quad [\because u^v = e^{v\log u}]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^{x\log x})$$

$$= e^{x\log x}.\frac{d}{dx}x\log x$$

$$= e^{x\log x}\left(\log x.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log x)\right)$$

$$= e^{x\log x}\left[(\log x\times 1)+x\frac{1}{x}\right]$$

$$= x^x[\log x+1] = x^x(1+\log x)$$

Remark. $\dfrac{d}{dx}(x^x) = x\,(1+\log x)$ **is considered as standard result.**

(ii) Let $\quad y = x^{e^x}$,

Then, taking logarithms of both the sides we get,

$$\log y = e^x\log x$$

Differentiating both the sides we get

$$\frac{dy}{dx}(\log y) = \frac{d}{dx}(e^x.\log x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log x\,.\,\frac{d}{dx}(e^x)+e^x.\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log x.e^x + e^x.\frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(e^x.\log x+\frac{e^x}{x}\right)$$

$$= x^{e^x}.e^x\left(\log x+\frac{1}{x}\right)$$

(iii) Let $\qquad y = x^{\sqrt{x}}$

Then, taking logarithms of both the sides we get,

$$\log y = \sqrt{x}\ \log x$$

Differentiating both the sides we get,

$$\frac{d}{dx}(\log y) = \log x.\frac{d}{dx}(\sqrt{x})+\sqrt{x}.\frac{d}{dx}(\log x)$$

$$\frac{1}{y}.\frac{dy}{dx} = \log x.\frac{1}{2}.x^{-1/2}+\sqrt{x}.\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2\sqrt{x}}.\log x+\frac{1}{\sqrt{x}}\right)$$

$$\therefore \quad \frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{\log x+2}{2\sqrt{x}}\right)$$

(iv) Let $\qquad y = x^{x^x}$

Then, taking logarithms of both the sides we get,

$$\log y = x^x \log x$$

Differentiating both the sides *w.r.t.* we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x^x.\log x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log x.\frac{d}{dx}(x^x)+x^x\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log x\left[x^x(1+\log x)\right]+x^x\times\frac{1}{x}$$

$$\frac{dy}{dx} = y[x^x.\log x(1+\log x)+x^{x-1}] \qquad \left[\because \frac{d}{dx}(x^x)=x^x(1+\log x)\right]$$

$$= x^{x^x}\left[x^x.\log x(1+\log x)+x^{x-1}\right]$$

EXERCISE (D)

1. Find the derivative of the following of functions :

(i) $x^{1/x}$ (ii) x^{x^2} (iii) $x^{\log x}$ (iv) $x^{\log(\log x)}$

(v) $(1+x)^{2x}$ (vi) $(\log x)^x$ (vii) x^{x^x} (viii) $(x^x)\sqrt{x}$

Ans. [(i) $x^{1/x}\left[\frac{1-\log x}{x^2}\right]$ (ii) $x^{x^2+1}(1+2\log x)$

(iii) $2x^{\log(x-1)}.\log x$ (iv) $x^{\log(\log x)}\left(\frac{1+\log(\log x)}{x}\right)$

(v) $2(1+x)^{2x}\left[\frac{x}{x+1}+\log(x+1)\right]$ (vi) $(\log x)^x(\log x+1)$

(vii) $x^{x^x}\left[x^x\log x(1+\log x)+x^{x-1}\right]$ (viii) $x^{x+\frac{1}{2}}\left[\left(\frac{2x+1}{2x}\right)+\log x\right]$]

2. Find $\frac{dy}{dx}$ when

(i) $y=\sqrt[3]{\frac{4+5x}{4-5x}}$ (ii) $y=e^{\frac{x^2}{1+x^3}}$ (iii) $y=x^x+x^{1/x}$

(iv) $y=2^x.\log x$ (v) $y=x^2 5^{3x}$ (vi) $y=\frac{1+\sqrt{x}}{1-\sqrt{x}}$

(vii) $y=\log\left[e^x.\frac{(x-2)^{3/4}}{x+2}\right]$

Ans. [(i) $\frac{40}{3(16-25x^2)}.\sqrt{\frac{4+5x}{4-5x}}$ (ii) $\frac{x(2-x^3)}{(1+x^3)^2}e^{\frac{x^2}{1+x^3}}$

(iii) $x^x(1+\log x)+x^{1/x}\left(\frac{1-\log x}{x^2}\right)$ (iv) $2^x\left[\log x.\log 2+\frac{1}{x}\right]$

(v) $5^{3x}.x(3x\log 5+2)$ (vi) $\frac{1}{\sqrt{x}(1-\sqrt{x})^2}$ (vii) $\frac{x^2-1}{x^2-4}$]

9. DIFFERENTIATION OF IMPLICIT FUNCTIONS

We often found relation between two variables x and y given by an equation of the form $f(x, y) = 0$ and in such case non of the variables is directly expressed in terms of others. But the equation may be defined y as a function of x. In such case, y is called an **implicit function** of x. So the relation is referred as Implicit function.

Thus, an implicit function we mean a function of x say y, where y can not be expressed in terms of x only. The examples of such functions are :

(i) $x^2-y^2+3x=5y$ (ii) $x^2+y^3-y^2+xy=3$ (iii) $x^2+xy^2-y=5$

In such a type of function, differential coefficients i.e. $\frac{dy}{dx}$ is to be found by differentiating the given equation term by term as per the following steps.

Steps :

1. Differentiate both the sides of the given relation with respect to x. While differentiating the y terms not containing x multiply the same coefficient by $\frac{dy}{dx}$. Thus y^2, will be differentiated as $2y\frac{dy}{dx}$ and $3y^5$ as $15y^4 \cdot \frac{dy}{dx}$.

2. Transfer all the terms containing $\frac{dy}{dx}$ to one side, and the terms not containing $\frac{dy}{dx}$ to the other side of the equation.

3. Divide both the sides by the coefficient of $\frac{dy}{dx}$, and get $\frac{dy}{dx}$ thereby.

4. Simplify the result by using the given relation if possible.

The following examples show how the implicit functions are differentiated.

ILLUSTRATION 10. Find $\frac{dy}{dx}$ of the following implicit functions :

(i) $x^2 - y^2 + 3x = 4y$ (ii) $x^3 + y^3 - y^2 + xy = 3$

SOLUTION

(i) Given, $x^2 - y^2 + 3x = 4y$

Differentiating both the sides w.r.t. x, we get,

$$\frac{d}{dx}(x^2 - y^2 + 3x) = \frac{d}{dx}4y$$

$$\Rightarrow \quad \frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) + 3\frac{d}{dx}(x) = 4\frac{d}{dx}y$$

$$\Rightarrow \quad 2x - 2y\frac{dy}{dx} + 3 = 4\frac{dy}{dx}$$

On transposition, we get,

$$\frac{dy}{dx}(2y+4) = (2x+3)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x+3}{2y+4}$$

It is not possible to simplify $\left(\frac{2x+3}{2y+4}\right)$ any further by using the given relation, $x^2 - y^2 + 3x$

Hence, $\frac{dy}{dx} = \frac{2x+3}{2y+4}$

(ii) Given, $x^3 + y^3 - y^2 + xy = 3$

Differentiating both the sides w.r.t. x we get,

$$\frac{d}{dx}(x^3 + y^3 - y^2 + xy) = \frac{d}{dx}(3)$$

$$\Rightarrow \quad \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(y^2) + \frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} + y\frac{d}{dx}(x) + x\frac{dy}{dx} = 0$$

$$= \quad 3x^2 + 3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

On transposition we get,

$$(3y^2 - 2y + x)\frac{dy}{dx} = -(3x^2 + y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(3x^2 + y)}{3y^2 - 2y + x)}$$

It is not possible to simplify $\frac{-(3x^2 + y)}{3y^2 - 2y + x)}$, any further by using the given relation i.e., $x^3 + y^3 - y^2 + xy = 3$

Hence, $\frac{dy}{dx} = \frac{-(3x^2 + y)}{3y^2 - 2y + x)}$

Note. In the above, $\frac{d}{dx}(xy)$ has been differentiated by the product rule i.e.,

$$\frac{d}{dx}(uv) = v\frac{d}{dx}(u) + u\frac{d}{dx}(v) = y\frac{d}{dx}(x) + x\frac{d}{dx}(y)$$

10. DIFFERENTIATION OF HIGHER ORDER

The differentiation of a function of x discussed so far in terms of $\frac{dy}{dx}$ is a differentiation of the first order. When the differentiation is carried out successively to any higher order viz. $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4} \ldots \frac{d^ny}{dx^n}$, it is called a **differentiation of higher order.** Finding of any higher order differentiation of any function say x, is possible because the derivative of a function is also a function of x. Thus, when $y = x^7$ and $\frac{dy}{dx} = 7x^6$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(7x^6) = 7\frac{d}{dx}(x^6) = 42x^5$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(42x^5) = 42\frac{d}{dx}x^5 = 210x^4$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d}{dx}(210x^4) = \left(210\frac{d}{dx}x^4\right) = 840x^3, \text{ and so on}$$

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Notation

If y denotes the function of x, then the first order derivative is denoted by $\frac{dy}{dx}$, or Dy, or fy, the Second **order derivation** is denoted by $\frac{d^2y}{dx^2}$, or D^2y, or $f''y$

$$\left[\because \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}\right]$$

The **Third order derivative** is denoted by $\frac{d^3y}{dx^3}$ or D^3y or f^3y

$$\left[\because \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}\right]$$

And the **Nth order of derivative** is denoted by $\frac{d^ny}{dx^n}$ or D^ny or f^ny

The concept of higher order differentiation is highly useful in the problems of optimisation.

The concept of higher order differentiation is highly useful in the problems of optimisation. The following illustrations show the procedure of computation of certain higher order differentiation.

The following illustrations show the procedure of computation of certain higher order differential coefficients w.r.t x when,

ILLUSTRATION 11. Find the 2nd and 3rd order differential coefficients w.r.t x when,

(i) $y=3x^3-9x$ (ii) $y=x^2\log x$

SOLUTION

Computation of the 2nd and 3rd order differential coefficients.

(i) When $y = 3x^3-9x$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^3-9x)=\frac{d}{dx}(3x^3)-\frac{d}{dx}(9x)=9x^2-9$$

The 2nd order differentiation is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(9x^2-9)=18x$$

The 3rd order differentiation is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}(18x)=18$$

(ii) When $y=x^2\log x$,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2\log x)=x^2\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x^2)$$

$$= x^2.\frac{1}{x}+\log x.2x=x+2x\log x=x(1+2\log x)$$

The 2nd order differential coefficient is given by,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}[x(1+2\log x)]$$

$$= x\frac{d}{dx}(1+2\log x)+(1+2\log x)\frac{d}{dx}(x)$$

$$= x\left(\frac{2}{x}\right)+(1+2\log x)(1)=2+1+2\log x=3+2\log x$$

The 3rd order differential coefficient is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}(3+2\log x)=\frac{2}{x}=2x^{-1}$$

ILLUSTRATION 12. Determine the 4th order derivative of the function,

$$y = \log\sqrt{3x+4}$$

Given,

$$y = \log(3x+4)^{\frac{1}{2}}=\frac{1}{2}\log(3x+4)$$

$\Rightarrow$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}\log(3x+4)\right]$$

$$= \frac{1}{2}\frac{d}{dx}\log(3x+4)=\frac{1}{2}.\frac{1}{3x+4}\frac{d}{x}(3x+4)$$

$$= \frac{1}{2(3x+4)}.3=\frac{3}{2(3x+4)}=\frac{3}{2}(3x+4)^{-1}$$

The 2nd order differential coefficient is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left[\frac{3}{2}(3x+4)^{-1}\right]=\frac{3}{2}\frac{d}{dx}[(3x+4)^{-1}]$$

$$= \frac{3}{2}(-1)(3x+4)^{-2}\frac{d}{dx}(3x+4)=\frac{-3}{2}(3x+4)^{-2}.3$$

$$= \frac{-9}{2}(3x+4)^{-2}$$

The 3rd order differential coefficient is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left[\frac{-9}{2}(3x+4)^{-2}\right]$$

$$= -\frac{9}{2}\frac{d}{dx}[(3x+4)^{-2}]=\frac{-9}{2}(-2)(3x+4)^{-3}\frac{d}{dx}(3x+4)$$

$$= 9(3x+4)^{-3}(3)=27(3x+4)^{-3}$$

Again, the 4th order differential coefficient is given by

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

$$= \frac{d}{dx}[27(3x+4)^{-3}]=27\frac{d}{dx}[(3x+4)^{-3}]$$

$$= 27(-3)(3x+4)^{-4}.\frac{d}{dx}(3x+4) = -81(3x+4)^{-4}(3)$$

$$= -243.(3x+4)^{-4} = \frac{-243}{(3x+4)^4}$$

EXERCISE (E)

1. Find $\frac{dy}{dx}$ from each of the following :

(i) $x^2+y^2=a^2$ (ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (iii) $x^3+3x^2y+y^3=a^3$

(iv) $x^2-2xy+y^2-2x=0$ (v) $x^4+x^2y^2+y^4=0$

(vi) $x^{2/3}+y^{2/3}=a^{2/3}$ (vii) $y=x^y$

Ans. $\Big[$ (i) $-\frac{x}{y}$, (ii) $-\frac{b^2x}{a^2y}$, (iii) $-\frac{x(x+2y)}{x^2+y^2}$ (iv) $\frac{y-x+1}{y-x}$,

(v) $-\frac{x(2x^2+y^2)}{y(x^2+2y^2)}$, (vi) $-\left(\frac{y}{x}\right)^{\frac{1}{3}}$ (vii) $\frac{y^2}{x(1-y\log x)}\Big]$

2. (i) If $y=\frac{x}{\sqrt{1+x^2}}$, Prove that $x^3\frac{dy}{dx}=y^3$

(ii) If $y(\log x+1)=x$, Prove that $\frac{dy}{dx}=\frac{\log x}{(\log x)^2}$

(iii) If $y=3\log(x+\sqrt{x^2-a^2})$, find $\frac{dy}{dx}$

(iv) If $y\sqrt{x^2+1}=\log(x+\sqrt{x+1})$, Prove that $(x^2+1)\frac{dy}{dx}+xy=1$

3. Find $\frac{d^2y}{dx^2}$ when

(i) $y=x^5$ (ii) $y=2.e^{5x}$ (iii) $y=3\log x$

(iv) $y=5^{2x}$ (v) $y=x^2\log x$ (vi) $y=\frac{2x+5}{x^2+5x+6}$

(vii) $y=\frac{1}{(x^2+a^2)(x^2+b^2)}$ (viii) $y=\log\log x$

Ans. $\Big[$ (i) $20x^3$, (ii) $50e^{5x}$, (iii) $-\frac{3}{x^2}$,

(iv) $4(\log 5)^2.5^{2x}$ (v) $2\log x+3$, (vi) $\frac{2}{(x+2)^3}+\frac{2}{(x+3^2}$

(vii) $\frac{2}{a^2-b^2}\left[\frac{3x^2-b^2}{x^2+b^2}-\frac{3x^2-a^2}{(x^2+a^2)}\right]$ (viii) $-\frac{1+\log x}{(x\log x)^2}\Big]$

11. APPLICATION IN ECONOMICS

1. Marginal, Average, And Total Functions

By a marginal function we mean an equation that expresses the variation on the margin, *i.e.* for a very small variation of x from a given value of x. Therefore, the marginal cost or revenue at a certain level of output is the change in the cost, or revenue that results when the output is increased by a very small amount from the level.

When, Total Cost = C and no. of Goods produced = x

The **Marginal Cost (MC)** = $\frac{dC}{dx}$

Similarly,

When, total Revenue = R and No. of Goods Sold = x

The **Marginal Revenue (MR)** = $\frac{dR}{dx}$

When, Total Utility = U and no. of goods consumed = x

The **Marginal Utility (MU)** = $\frac{dU}{dx}$

When, Total Product = Q and No. of Labour = L

The **Marginal Product** = $\frac{dQ}{dL}$

[**Note.** Where $C=f(x)$ that gives the total cost, and $R=f(x)$, that gives the total revenue].

By an **average function,** on the other hand, we mean an equation that expresses the variation of one quantity, say y over a specified range of values of another quantity, say x *i.e.* from a given value to a certain selected value, say from x to $x+\delta x$, or say from 10 to 12 (where $x=10$, and $\delta x=2$).

Thus, an average cost (or revenue) is the ratio between the total cost (revenue), and the whole of the output concerned.

Thus, Average Cost is obtained by AC, or $y=\frac{C}{x}$

And Average Revenue is obtained by AR, or $y=\frac{R}{x}$

[where $C=f(x)$, and $R=f(x)$, and x = quantity of the product]

Further, by a total function we mean a function of the output (*i.e.* x).

Thus, the total cost is obtained by $C=f(x)$, or $=c.x$

And the total revenue is obtained by $R=f(x)$, or px

[Where c = cost per unit, p = price unit and x = quantity of a product.]

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EXAMPLE 1. If total cost function $C = 80x - 15x^2 + x^3$ find out (i) Marginal cost function, (ii) Average cost function, and (iii) Output at which Average Cost = Marginal Cost

SOLUTION

(i) Given, total cost function, $C = 80x - 15x^2 + x^3$

Marginal Cost function or $MC = \frac{dC}{dx} = \frac{d}{dx}(80x - 15x + x^3)$

Thus, $MC = 80 - 30x + 3x^2$

(ii) Given, $C = 80x - 15x^2 + x^3$ and No. of units $= x$

$$\text{Average cost function(AC)} = \frac{\text{Total Cost}}{\text{No.of units}} = \frac{80x - 15x^2 + x^3}{x} = 80 - 15x + x^2$$

Thus, $AC = 80 - 15x + x^2$

(iii) Output at which AC is minimum when, $\frac{d(AC)}{dx} = 0$

we have, $AC = 80 - 15x + x^2$

when, $\frac{d(AC)}{dx} = 0, \Rightarrow -15 + 2x = 0$, or $x = 7.5$

Now, $AC = 80 - 15x + x^2 = 80 - (15 \times 7.5) + (7.5)^2 = ₹\, 23.75$

$MC = 80 - 30x + 3x^2 = 80\ (30 \times 7.5) + 3\ (7.5)^2 = ₹\, 23.75$

Thus, at output 7.5, the MC = AC

EXAMPLE 2. Find the Marginal Revenue Function, when the Demand Function is

$$Q_d = x^2 + 2x + 5$$

Calculate the Marginal Revenue at quantity demanded at $x = 4$ and $x = 10$.

SOLUTION

Demand Function is also otherwise known as Average Revenue.

Thus, Average Revenue or $AR = x^2 + 2x + 5$

Then, Total Revenue, R = AR × Quantity of emended or $R = (x^2 + 2x + 5) \times x = x^3 + 2x^2 + 5x$

$$\text{Marginal Revenue (MR)} = \frac{d(R)}{dx} = \frac{d}{dx}(x^3 + 2x^2 + 5x) = 3x^2 + 4x + 5$$

$\therefore$ $MR = 3x^2 + 4x + 5$

When $x = 4$, $MR = 3 \times (4)^2 + 4 \times 4 + 5 = 69$

When $x = 10$, $MR = 3\ (10)^2 + 4 \times 10 + 5 = 345$

EXAMPLE 3. Find the Marginal Cost function when the Average Cost function,

$$AC = 18x + 8 + \frac{45}{x}$$

SOLUTION

When, Average Cost or $AC = 18x + 8 + \frac{45}{x}$, $C = AC \times x = (18x + 8 + \frac{45}{x})x$

or Total cost or $C = 18x^2 + 8x + 45$

$$\text{Marginal Cost}\ ,\quad MC = \frac{dC}{dx} = \frac{d}{dx}\left(18x^2 + 8x + 5\right) = 36x + 8$$

ILLUSTRATION 13. If the total cost function is given by $C = 0.1x^2 + 5$, then find, (*i*) the marginal function, (*ii*) average function, and (*iii*) the marginal cost where 4 units are produced. Also, interpret the result.

SOLUTION

(*i*) Marginal Cost

This is given by $MC = \frac{dC}{dx} = \frac{d}{dx}(0.1x^2 + 5) = 0.2x$

(*ii*) Average cost

This is given by $AC = \frac{C}{x} = \frac{0.1x^2 + 5}{x} = 0.1x + \frac{5}{x}$

(*iii*) Marginal Cost when 4 units are produced

Under the above (*i*) we have $MC = 0.2x$

Now, when $x = 4$, $MC = 0.2(4) = 0.80$

Interpretation

The above result shows that if the production is increased by 1 unit (*i.e.* from 4 units to 5 units) then the cost of additional units would be 0.80. It may be noted that the actual cost of producing one more unit beyond 4 units is $C(5) - C(4) = [0.1(5)^2 + 5] - [0.1(4)^2 + 5] = 7.5 - 6.6 = 0.90$.

ILLUSTRATION 14. Determine (*i*) the total revenue function, and (*ii*) the marginal revenue function for a firm operating under perfect competition with a current price of ₹ 2 per unit.

SOLUTION

Given $p = 2$

(*i*) Total Revenue function

This is obtained by $R = f(x) = p.x = 2x$

(*ii*) Marginal Revenue function

This is given by $MR = \frac{dR}{dx} = \frac{d(2x)}{dx} = 2$

The above result shows that the marginal revenue is ₹ 2 irrespective of the number of units sold.

ILLUSTRATION 15. A firm with a linear demand function can sell 1000 units when the price is ₹ 4 per unit, and 1500 units when the price is ₹ 2 per unit.

On the above premises, determine, (i) the demand function, (ii) the total revenue function, (iii) the average revenue function, and (iv) the marginal revenue function.

SOLUTION

(*i*) Demand Function

Let the demand function be $p = a + bx$ [$\because$ it is linear]

Where p is the price, and x is the quantity demanded at this price.

Given $x = 1000$,

when $p = 4$

$x = 1500$,

when $p = 2$

Thus we have, $4 = a + 1000b$ ...(1)

And $2 = a + 1500b$

Solving the above two equations simultaneously we get, $a = 8$, $b = -1/250$.

$\therefore$ The required demand function is $p = 8 - \frac{x}{250}$

(ii) Total Revenue Function

This is obtained by $R = f(x) = px = \left(8 - \frac{x}{250}\right)x = 8x - \frac{x^2}{250}$

(iii) Average Revenue Function

This is given by $$AR = \frac{R}{x} = \frac{8x - \frac{x^2}{250}}{x} = 8 - \frac{x}{250}$$

(iv) Marginal Revenue Function

This is obtained by

$$MR = \frac{dR}{dx} = \frac{d}{dx}\left(8x - \frac{x^2}{250}\right)$$

$$= d\frac{d}{dx}(8x) - \frac{1}{250}\frac{d}{dx}(x^2)$$

$$= 8 - \frac{1}{250}.2x = 8 - \frac{x}{125}$$

ILLUSTRATION 16. If the total cost function is given by $C = a + bx + cx^2$, then show that $\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$, where x is the output, MC = marginal cost and AC is the average cost.

SOLUTION

Given, $$C = a + bx + cx^2$$

We have $$AC = \frac{C}{x} = \frac{a + bx + cx^2}{x} = \frac{a}{x} + b + cx$$

And $$MC = \frac{dC}{dx} = \frac{d}{dx}(a + bx + cx^2) = b + 2cx$$

Thus, $$\frac{1}{x}(MC - AC) = \frac{1}{x}[(b + 2cx) - \left(\frac{a}{x} + b + cx\right)$$

$$= \frac{1}{x}\left(cx - \frac{a}{x}\right) = c - \frac{a}{x^2}$$

And $$\frac{d}{dx}(AC) = \frac{d}{dx}\left(\frac{a}{x} + b + cx\right) = \frac{d}{dx}(ax^{-1}) + \frac{d}{dx}(b) + \frac{d}{dx}(cx)$$

$$= -ax^{-2} + 0 + cx^{1-1} = \frac{-a}{x^2} + c \text{ or } c - \frac{a}{x^2}$$

From the above two equations it is found that

$$\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC). \quad \text{Hence proved.}$$

ILLUSTRATION 17. A producer finds that t employees will produce a total of x units of product per day, where $x = 2t$. If the demand equation for the product is $p = -0.5x+20$, then find the marginal revenue product when $t = 5$. Also, interpret your result.

SOLUTION

The total revenue is given by $R = px$

$$= (-0.5x + 20)\,x = -0.5x^2 + 20x$$

$$= -0.5\,(2t)^2 + 20\,(2t) = -0.5 \times 4t^2 + 40t = -2t^2 + 40t$$

[$\because x = 2t$ given]

Thus, the marginal revenue product is given by

$$MR = \frac{dR}{dt}$$

$$= \frac{d}{dt}(-2t^2 + 40t) = -4t + 40$$

Now, when $t = 5$, $$\frac{dR}{dt} = -4(5) + 40 = 20.$$

Interpretation

The above result shows that if the 6th employee is hired, then the extra revenue generated would be approximately 20.

ILLUSTRATION 18. The average cost function for a commodity is $AC = x + 5 + \frac{36}{x}$, where x is the output. Determine the output for which the AC is decreasing with the increase in the output. Also, find the total cost, and the marginal cost as the function of x.

SOLUTION

Given $$AC = x+5+\frac{36}{x}$$

Thus, $$\frac{d}{dx}(AC) = \frac{d}{dx}\left(x+5+\frac{36}{x}\right) = 1 - \frac{36}{x^2}$$

For AC to be increasing, we have $\frac{d}{dx}(AC) > 0$

$\Rightarrow 1 - \frac{36}{x^2} > 0,\ 1 > \frac{36}{x^2}$ or $\Rightarrow x^2 > 36, \Rightarrow x > 6$ [$\because$ x is a quantity and can not be –ve]

Similarly, for AC to be decreasing we have,

$\frac{d}{dx}(AC) < 0, \Rightarrow 1 - \frac{36}{x^2} < 0$, or $1 < \frac{36}{x^2} \Rightarrow x^2 < 36, \Rightarrow 0 < x < 6$

For finding the total cost function C where $AC = \frac{C}{x}$

$$\Rightarrow \qquad C = x(AC) = x\left[x + 5 + \frac{36}{x}\right] = x^2 + 5x + 36$$

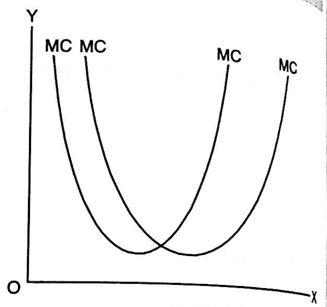
Again, to find the maginal cost we have,

$$MC = \frac{dC}{dx} = \frac{d}{dx}x^2 + 5x + 36 = 2x + 5$$

RELATIONSHIP AMONG THE TOTAL, MARGINAL & AVERAGE FUNCTIONS

A. In terms of Cost

In terms of cost, the total cost function, including the fixed cost, is represented by TC $= f(x) + b$ [where b is the fixed cost] From the above TC function, the other cost functions can be derived as under :



Average total cost, or ATC $= \dfrac{f(x) + b}{x}$

Average variable cost or AVC $= \dfrac{f(x)}{x}$

Average fixed cost, or AFC $= \dfrac{b}{x}$

Thus, ATC = AFC + AVC $= \dfrac{f(x)}{x} + \dfrac{b}{x} = \dfrac{f(x)+b}{x}$

And Marginal cost, or MC $= \dfrac{dC}{dx}$,

Although, cost curves take many different forms under different situations, yet under the natural economic limitations, the marginal and the average cost curves usually take U shapes shown as under:

The relation between the MC and AC may be drawn as follows :

(*i*) When AC is decreasing, MC < AC
(*ii*) When AC is minimum, AC = MC
(*iii*) When AC is increasing MC > AC

Proof.

The above relations between the MC and AC is derived through calculus as follows:

We have, $\quad AC = \dfrac{\text{Total Cost}}{\text{No. of Units}} = \dfrac{C}{x}$, and MC $= \dfrac{d}{dx}(C)$

Differentiating the AC we have,

$$\frac{d}{dx}(AC) = \frac{d}{dx}\left(\frac{C}{x}\right) = \frac{x\frac{d}{dx}C - C.\frac{d}{dx}(x)}{x^2} = \frac{1}{x^2}\left(x.\frac{dC}{dx} - C\right) = \frac{1}{x}\left(\frac{dC}{dx} - \frac{C}{x}\right) = \frac{1}{x}\,(MC - AC)$$

From the above, we may now conclude as follows :

(*i*) When AC curve slopes downwards, its slope will be –ve. In other words,

$$\frac{d}{dx}\left(\frac{C}{x}\right) < 0, \quad \Rightarrow MC - AC < 0, \quad \Rightarrow MC < AC$$

Thus, when AC curve is declining, MC curve will be below AC.

(*ii*) When AC curve reaches a minimum point, its slope becomes zero, *i.e.* $= \dfrac{d}{dx}\left(\dfrac{C}{x}\right) = 0, \Rightarrow$ MC – AC = 0, ⇒ MC = AC. Thus, **MC and AC curves intersect at the point of minimum average cost.**

(*iii*) When AC curve rises upwards, its slope is +ve. In other words, $\dfrac{d}{dx}\left(\dfrac{C}{x}\right) > 0, \Rightarrow MC > AC$.

Thus, when AC curve slopes upwards, MC curve will be above AC curve.

B. In terms of revenue

In terms of revenue, the total revenue function is represented by

$$TR = R = xp = x f(x) \qquad [\because p = f(x)]$$

From the above total revenue functions, the other revenue functions can be derived as under:

Average Revenue Function, or AR $= \dfrac{xp}{x}$, or $\dfrac{xf(x)}{x}$

And Marginal Revenue Function, or MR $= \dfrac{dR}{dx} = p + x\left(\dfrac{dp}{dx}\right)$

If at any level of x, MR lies between AR then,

$$\left(p + x\frac{dp}{dx}\right) < p \qquad [\because AR = p]$$

$$\therefore \qquad x\frac{dp}{dx} < 0, \text{ or } \frac{dp}{dx} < 0 \qquad [\because x \not< 0]$$

$$\Rightarrow \qquad \frac{d}{dx}(AR) < 0$$

i.e. if MR is to be between AR, then $\dfrac{d}{dx}$(AR) < 0, or AR curve should be declining.

ILLUSTRATION 19. The total cost C for the output x is given by the function, C$= \dfrac{2}{3}x + \dfrac{35}{2}$.

Determine (*i*) the cost when the output is 5 units.
(*ii*) the average cost when the output is 15 units.
And (*iii*) the marginal cost when the output is 2 units.

SOLUTION

(*i*) Cost :

Given $\qquad C = \dfrac{2}{3}x + \dfrac{35}{2}$

∴ When output is 5 units, C $= \dfrac{2}{3}(5) + \dfrac{35}{2} = \dfrac{10}{3} + 17.5 = 20.83$

(*ii*) Average Cost (AC) :

When output is 15 units, C $= \dfrac{2}{3}(15) + \dfrac{35}{2} = 10 + 17.5 = 27.5$

$\therefore$ Average Cost (AC) $\dfrac{\text{Total Cost}}{\text{No. of Units}} = \dfrac{C}{x} = \dfrac{27.5}{15} = 1.83$

***(iii)* Marginal Cost (MC) :**

We have $\quad MC = \dfrac{dC}{dx} = \dfrac{d}{dx}\left(\dfrac{2}{3}x + \dfrac{35}{2}\right) = \dfrac{2}{3}$

Note. MC remains the same regardless of the number of outputs produced.

ILLUSTRATION 20. Examine the marginal and average cost relations when the total variable cost is $C = x^3 - 3x^2 + 15x$.

SOLUTION

Given, total variable Cost or $C = x^3 - 3x^2 + 15x$

We have, Average Variable Cost or AVC $= \dfrac{C}{x} = \dfrac{x^3 - 3x^2 + 15x}{x} = x^2 - 3x + 15$

And Marginal Cost or MC $= \dfrac{dC}{dx} = \dfrac{d}{dx}(x^3 - 3x^2 + 15x) = 3x^2 - 6x + 15$

Examination of the relation between the MC and AVC

(i) When AVC declines, we have, $\dfrac{d}{dx}(x^2 - 3x + 15) < 0$, or $2x - 3 < 0$ ...(1)

But when AC declines, MC < AC

$\Rightarrow \quad 3x^2 - 6x + 15 < x^2 - 3x + 15$

$\therefore \quad 2x^2 - 3x < 0, \Rightarrow x(2x - 3) < 0$, But $x > 0$ [$\because$ it is a quantity]

$\therefore \quad 2x - 3 < 0$ ...(2)

$\therefore$ The inequality (2) must hold good if MC is to lie below AC, and the same inequality (1) suggests that AC is declining.

(ii) When AVC rises, we have $\dfrac{d}{dx}(x^2 - 3x + 15) > 0, \Rightarrow 2x - 3 > 0$ ...(3)

But when AC rises, MC should be above AC

$\Rightarrow \quad 3x^2 - 6x + 15 > x^2 - 3x + 15$

$\Rightarrow \quad 2x^2 - 3x > 0$ or $x(2x - 3) > 0 \therefore x > 0$, and ...(4)

We have, $2x - 3 > 0$, which again is the same inequality as in (3)

(iii) When AVC is minimum we have, $\dfrac{d}{dx}(x^2 - 3x + 15) = 0$,

Thus, $2x - 3 + 0 = 0$ or $\quad 2x - 3 = 0$ ...(5)

But when AC is minimum, AC = MC

$\therefore \quad 3x^2 - 6x + 15 = x^2 - 3x + 15, \quad \Rightarrow 2x^2 - 3x = 0,$

$\Rightarrow \quad x(2x - 3) = 0, \Rightarrow 2x - 3 = 0$

Which is the same result as in (5) above.

ILLUSTRATION 21. The total cost function of a firm is given by
$C = 0.04x^3 - 0.9x^2 + 10x + 10.$

Find (i) AC, (ii) MC, (iii) Slope of AC (iv) Slope of MC, and (v) Value of x at which the average variable cost is the minimum.

SOLUTION

Given, $\quad C = 0.04x^3 - 0.9x^2 + 10x + 10$

***(i)* Average Cost (AC) :** We have,

$AC = \dfrac{C}{x} = \dfrac{0.04x^3 - 0.9x^2 + 10x + 10}{x} = 0.04x^2 - 0.9x + 10 + \dfrac{10}{x}$

***(ii)* Marginal Cost (MC) :** We have

$MC = \dfrac{dC}{dx} = \dfrac{d}{dx}\, 0.04x^3 - 0.9x^2 + 10x + 10 = 0.12x^2 - 1.8x + 10$

***(iii)* Slope of Average Cost (AC) :**

This is given by $\dfrac{d}{dx}(AC) = \dfrac{d}{dx}\left(0.04x^2 - 0.9x + 10 + \dfrac{10}{x}\right) = 0.08x - 0.9 - \dfrac{10}{x^2}$

$= \dfrac{1}{x}\left(0.08x^2 - 0.9x - \dfrac{10}{x}\right) = \dfrac{1}{x}\left[(0.12x^2 - 0.04x^2) - 1.8x + 0.9x + 10 - 10 - \dfrac{10}{x}\right]$

$= \dfrac{1}{x}\left[(0.12x^2 - 1.8x + 10) - (0.04x^2 - 0.9x + 10 + \dfrac{10}{x})\right] = \dfrac{1}{x}(MC - AC)$

***(iv)* Slope of Marginal Cost (MC)**

This is given by $\dfrac{d}{dx}(MC) = \dfrac{d}{dx}(0.12x^2 - 1.8x + 10) = 0.24x - 1.8$

***(v)* Value of x when Average Variable Cost (AVC) is minimum**

When AVC is minimum, the slope of AVC curve is zero.

i.e. $\quad \dfrac{d}{dx}\left(\dfrac{C}{x}\right) = 0, \ \dfrac{d}{dx}\left(0.04x^2 - 0.9x + 10 + \dfrac{10}{x}\right) = 0 \Rightarrow 0.08x - 0.9 - \dfrac{10}{x^2} = 0.$

$\Rightarrow \quad 0.08x^2 - 0.9x - 10 = 0$

ILLUSTRATION 22. The cost function of producing x shoes is $6x + 5x^2 - 2x^3 - 4$. The total cost of producing a pair of shoes is ₹ 12. Set up the (i) Marginal Cost and (ii) the average cost function.

SOLUTION

Given, $\quad C = 6x + 5x^2 - 2x^3 - 4$

(i) Marginal Cost Function or MC $= \dfrac{d}{dx}(C) = \dfrac{d}{dx}(6x + 5x^2 - 2x^3 - 4) = 6 + 10x - 6x^2$

Since, the total cost function $C = 6x + 5x^2 - 2x^3 - 4$

(ii) The Average cost function is given by

$AC = \dfrac{C}{x} = \dfrac{6x + 5x^2 - 2x^3 - 4}{x} = 6 + 5x - 2x^2 - \dfrac{4}{x}$

$$\text{Or} \quad AC = 6 + 5x - 2x^2 - \frac{4}{x}$$

12. TAX YIELD

On most of the transactions, various types of taxes, viz. Sales tax (*i.e.* levied at a % of Sales Value), Excise duty (*i.e.* levied per unit) etc. are imposed by the competent governments for raising their revenue. Marketeers, in turn, shift the burden of such taxes on to the consumers by way of raising their prices. As such, taxes affect very much the optimum prices, and quantities of production and sales. The following illustrations will show how the various effects of taxes on the marketing are worked out through mathematical functions.

ILLUSTRATION 23. The demand and cost functions of a monopolist are respectively as follows:

$p = 20 - 4x$, and $\quad C = 4x$.

With reference to the above two functions determine,

(i) the optimum level of output and the price for profit maximization,
(ii) the new profit maximizing output, and the price if a tax of ₹ 0.50 per unit is imposed.
(iii) the output, and the price that correspond to the maximum profit, if a tax of t per unit is imposed.
(iv) the tax that maximizes the tax revenue, and the maximum tax revenue as well.
(v) the total tax revenue, if a sales tax of 10% is levied.

SOLUTION

(*i*) We have, Tax Revenue or $R = p\,x = (20 - 4x)\,x = 20x - 4x^2$

Total cost, or $\quad C = 4x$

And Profit, or $\quad P = R - C = 20x - 4x^2 - 4x = 16x - 4x^2$

Thus, $\quad \dfrac{dP}{dx} = \dfrac{d}{dx}(16x - 4x^2) = 16 - 8x$

If $\quad \dfrac{dP}{dx} = 0$, then $16 - 8x = 0, \Rightarrow x = 2$

Now, $\quad \dfrac{d^2P}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dP}{dx}\right) = \dfrac{d}{dx}(16 - 8x) = -8$, *i.e.* < 0

Thus, $x = 2$ gives maximum profit.

Putting x at 2 in the demand function, $P = 20 - 4x$ we get,

$p = 20 - 4(2) = 12$ units

Again, putting x at 2 in the profit function $P = 16x - 4x^2$

We get, $16(2) - 4(2)^2 = 32 - 16 = 16$

Hence, the required optimum level of output is 12 units and the price is ₹ 16.

(*ii*) With the imposition of a tax of ₹ 0.50 per unit, the total cost increases by $0.5x$.

Thus, the new cost function or

$C_N = 4x + 0.5\,x = 4.5x$

And the new profit function, or

$P_N = R - C_N = 20x - 4x^2 - 4.5x$

$= 15.5x - 4x^2$

Thus, $\quad \dfrac{dP_N}{dx} = \dfrac{d}{dx}(15.5x - 4x^2) = 15.5 - 8x$

Setting $\quad \dfrac{dP_N}{dx} = 0$, gives $15.5 - 8x = 0, \Rightarrow x = 1.94$ approx.

Again, $\quad \dfrac{d^2P_N}{dx} = \dfrac{d}{dx}(15.5 - 8x) = -8 < 0$

Hence, the new profit is maximum when $x = 1.94$

Putting x at 1.94 in the demand function, $p = 20 - 4x$ we get,

$p = 20 - 4\,(1.94) = 20 - 7.76 = 12.24$ units.

Putting x at 1.94 in the new profit function

$P_N = 20x - 4x^2 - 4.5x$ we get,

$P_N = 20(1.94) - 4(1.94)^2 - 4.5\,(1.94)$

$= 38.80 - 15.05 - 8.73 = 15$ approx.

Hence, the required new output level is 12.24 units, and the price is ₹ 15.

(*iii*) If a tax of t per unit is levied then we have,

$C_N = C + tx = 4x + tx$

And $\quad P_N = R - C_N = (20 - 4x^2) - (4x + tx) = 16x - tx - 4x^2$

Thus, $\quad \dfrac{dP_N}{dx} = \dfrac{d}{dx}(16x - tx - 4x^2) = 16 - t - 8$

Setting $\quad \dfrac{dP_N}{dx} = 0$ gives $16 - t - 8x = 0, \Rightarrow x = (16 - t)/8$

Since $\quad \dfrac{d^2P_N}{dx^2} = -8 < 0$, Profit is maximum when $x = (16 - t)/8$

∴ The corresponding price, or

$$p = 20 - 4\left(\frac{16 - t}{8}\right) = 20 - 8 + t/2$$

$$= 12 + t/2$$

(*iv*) Let a tax of t per unit of output maximizes the total tax revenue T with the output level x.

Then $\quad T = tx$

In the above, we have obtained that when a tax of t per unit is imposed, then $x = (16 - t)/8$

$$\therefore \quad T = t\frac{(16 - t)}{8} = \frac{16t - t^2}{8}$$

T will be maximum when $\dfrac{dT}{dt} = 0$ and $\dfrac{d^2T}{dt^2} < 0$

Setting $\quad \dfrac{dT}{dt} = 0$ gives $\dfrac{16 - 2t}{8} = 0$, or $t = 8$

Since $\frac{d^2T}{dt^2} = -\frac{1}{4} < 0$, a tax of 8 per unit maximizes the total tax revenue.

$\therefore$ The corresponding tax revenue or $T = \frac{16(8)-(8)^2}{8} = 8$

(v) If a sales tax of 10% is levied, then we have,

Total tax $= \frac{10}{100}(20x - 4x^2) = 2x - \frac{2}{5}x^2$

$\therefore \quad C_N = 4x + 2x - \frac{2}{5}x^2 = 6x - \frac{2}{5}x^2$

$\Rightarrow \quad P_N = R - C_N = 14x - \frac{18}{5}x^2$

$\therefore \quad \frac{dP_N}{dx} = 14 - \frac{36}{5}x$

Setting $\frac{dP_N}{dx} = 0$, and solving for x to get, $x = \frac{35}{18}$

Again $\frac{d^2P_N}{dx^2} = -\frac{36}{5} < 0$

Hence, the optimum output is $x = \frac{35}{18}$, and the total tax revenue

Or $TR = 2\left(\frac{35}{18}\right) - \frac{2}{5}\left(\frac{35}{18}\right)^2 = \frac{385}{162}$

ILLUSTRATION 24. *The total cost function of a firm is* $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$, *where C is the total cost and x is the output.*

A tax of ₹ 2 per unit is levied, and the producer adds this tax to his cost.

If the demand function is P = 2530 – 5x, where P is the price per unit, then find the profit maximizing output and the price.

SOLUTION

We have, Total Revenue or TR = (2530 – 5x) x = 2530 x – $5x^2$

After levy of a tax of ₹ 2 per unit the total cost function is given by

$$TC = \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x = \frac{1}{3}x^3 - 5x^2 + 30x + 10$$

Profit or $P = TR - TC = (2530x - 5x^2) - \left(\frac{1}{3}x^3 - 5x^2 + 30x + 10\right)$

$$= 2500x - \frac{1}{3}x^3 - 10$$

For maximization we have, $\frac{dP}{dx} = \frac{d}{dx}\left(2500x - \frac{1}{3}x^3 - 10\right) = 2500 - x^2$

Setting $\frac{dP}{dx} = 0$ gives $2500 - x^2 = 0 \Rightarrow x = 50$ (ignoring –ve value of x)

Again, $\frac{d^2P}{dP^2} = \frac{d}{dx}(2500 - x^2) = -2x$ *i.e.* < 0.

Putting x at 50 in the equation P = 2530 – 5x we get, P = 2530 – 5(50) = 2530 – 250 = 2280.

Hence, the required profit maximizing output is 50 units, and the price is ₹ 2280.

13. PRODUCTION FUNCTION

Production function refers to the functional relationship between quantity of the goods produced (outputs) and factors of production (inputs). Mathematically, such basic relationship between inputs and outputs may be expressed as

$Q = f(L, K, N)$, (Where, L = Labour, K = Capital and N = Land)

In economics, the **Cobb-Douglas** production function is a particular functional form of the production function widely used to represent the technological relationship between the amounts of two or more inputs particularly physical capital and labour, and the amount of output can be produced by those inputs.

Thus, $P = A.L^{\beta}.K^{\alpha}$

Where, P = Total production, L = Labour, K = Capital, A = Total factor productivity

α and β are output elastics of capital and labour. These values are constants and determined by available technology.

ILLUSTRATION 25. The production function of a firm is given by $P = 20.L^{1/3}.K^{2/3}$. Find the average and the marginal product, and evaluate them when L = 8, and K = 27. Also, interpret the result.

SOLUTION

(*i*) The average product of labour is obtained by

Average Production of Labour (APL) $= \frac{P}{L} = \frac{20L^{1/3}K^{2/3}}{L} = 20L^{-2/3}K^{2/3}$

Again, the average product of capital is obtained by

Average Production of Capital (APK) $= \frac{P}{K} = \frac{20L^{1/3}K^{2/3}}{K} = 20L^{1/3}K^{-1/3}$

When L = 8, and K = 27, we get,

$$APL = 20(8)^{-2/3}.(27)^{2/3} = \frac{20}{\sqrt[3]{8^2}}\sqrt[3]{27^2} = \frac{20}{4} \times 9 = 45$$

And

$$APK = 20(8)^{1/3}.(27)^{-1/3} = (20 \times 2)\frac{1}{\sqrt[3]{27}} \times \frac{1}{3} = \frac{40}{3}$$

Thus, when L = 8, and K = 27, then by holding K at 27, the output per unit of labour is 45 units. But if L is held at 8, the output per unit of capital is $\frac{40}{3}$ units.

(*ii*) The marginal product of labour is obtained by

Marginal Production of Labour (APL) $= \frac{dP}{dL} = \frac{20}{3}L^{-2/3}K^{2/3}$

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And the marginal product of capital is obtained by

Marginal Production of Capital (MPK) = $\frac{dP}{dK} = \frac{40}{3} L^{1/3} K^{-1/3}$

Where L = 8, and K = 27 we get,

$$MPL = \frac{20}{3}(8)^{-2/3} \cdot (27)^{2/3} = \frac{20}{3} \times \frac{1}{4} \times 9 = 15$$

$$MPK = \frac{40}{3}(8)^{1/3} \cdot (27)^{-1/3} = \frac{40}{3} \times 2 \times \frac{1}{3} = \frac{80}{9}$$

Thus, when L = 8, and K = 27, increasing L to 9 and holding K at 27 will increase the output by 15 units. But if K is increased to 28 while L is held at 8, the output increases by $\frac{80}{9}$ or 9 units approx.

ILLUSTRATION 26. The production function of a commodity is given by

$$P = 40q + 3q^2 - \frac{q^3}{3},$$

where P is the total output, and q is the units of input. Find (i) the number of units of input required to give the maximum output, and (ii) the maximum value of the marginal product.

Also, verify that the average product when maximum is equal to the marginal product.

SOLUTION

(i) Number of Units Required

Given, $P = 40q + 3q^2 - \frac{q^3}{3}$

We have marginal product, or

$$q = \frac{d}{dq}(40q - 3q^2 - q^3/3) = 40 - 6q - \frac{1}{3} \times 3q^2 = 40 - 6q - q^2$$

For maxima or minima we have, 40 + 6q – q2 = 0

⇒ $(q + 10)(q - 4) = 0, \Rightarrow q = -10$, or 4

Again, we have $\frac{d^2P}{dq^2} = 6 - 2q$

Thus, $\left[\frac{d^2P}{dq^2}\right]_{q=4} = 6 - 2(4) = -2$, *i.e.* < 0

And $\left[\frac{d^2P}{dq^2}\right]_{q=10} = 6 - 2(-10) = 26$, *i.e.* > 0

Hence, the output is maximum where 4 units of inputs are used.

(ii) Maximum value of the marginal Product

We have, $MP = \frac{dP}{dq} = 40 + 6q - q^2$

For maxima, or minima we have, $\frac{d(MP)}{dq} = 6 - 2q = 0$ or $2q = 6, \therefore q = 3$

$\therefore \quad q = 3$

Also, we have $\frac{d^2(MP)}{dq^2} = -2$, *i.e.* < 0.

∴ The maximum value of the marginal product = 3.

(iii) Verification

We have, average product, or

$$AP = \frac{P}{q} = \frac{40q + 3q^2 - \frac{1}{3}q^3}{q}$$

$$= 40 + 3q - q^{2/3}$$

For maxima or minima we have, $\frac{d^2(AP)}{dq} = 3 - \frac{2q}{3} = 0$ or $\frac{2q}{3} = 3, \therefore q = \frac{9}{2}$

Again, we have $\frac{d^2(AP)}{dq^2} = \frac{-2}{3}$, *i.e.* < 0

∴ The average product is maximum when $q = 9/2$

The average product $\left(\text{when } q = \frac{9}{2}\right) = 40 + 3\left(\frac{9}{2}\right) - \frac{81}{12} = \frac{187}{4}$

And the marginal product (when AP is maximum *i.e.* $q = 9/2$)

$$= 40 + 27 - \frac{81}{4} = \frac{187}{4} = 47 \text{ app.}$$

14. CONSTRAINED OPTIMIZATION PROBLEM

Constrained optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints of those variables. There are several types of constraints–primarily equality constraints, inequality constraints and integer constraints. The set of candidate solutions that satisfy all constraints is called the feasible set. If the constraint problem has only equality constraints, the method of **Lagrange multipliers** can be used to convert it into an unconstrained problems. But with the inequality constraints, the problem can be characterized in terms of the **'Geometric Optimality Conditions'.**

Most of the economic problems relating to maxima, and minima are concerned with the constrained optimization, where we are required to determine how much of x and y commodities should a consumer buy with a limited purchasing power to maximize his utility. The following illustrations will show how such economic problems are resolved:

ILLUSTRATION 27. Given the function, $U = x^2 + y^2 + w^2$ subject to the linear constraint, $y + x + w = 1$, find at what point, U has a maximum, or a minimum value. Also, determine the value of U (*i.e.* utility).

SOLUTION

Combining the function, and the constraint through Lagrange's multiplier λ we have,

$$Z = x^2 + y^2 + w^2 + \lambda\,(x + y + w - 1) = 0$$

Under the first order condition we have,

$$fx\left(\equiv \frac{\partial Z}{\partial x}\right) = 2x + \lambda = 0$$

$$fy\left(\equiv \frac{\partial Z}{\partial y}\right) = 2y + \lambda = 0$$

$$fw\left(\equiv \frac{\partial Z}{\partial w}\right) = 2w + \lambda = 0$$

And $$f\lambda\left(\equiv \frac{\partial Z}{\partial \lambda}\right) = x + y + w - 1 = 0$$

Solving the above equations we get, $x = y = w = \frac{1}{3}$, and $\lambda \equiv \frac{2}{3}$

In other words, the function U can have either a minimum number, or a maximum value at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. We apply the second order condition here to decide about this.

Under the second order condition we have,

Bordered Hessian $|\overline{H}| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$

Thus, the principal minors are : $|\overline{H}_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -4 < 0$, and

$$|\overline{H}_3| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -12 < 0$$

Thus, all the principal minors are < 0.

$\therefore\ d^2Z$ will have +ve value.

Since d^2Z possesses a +ve value, the given function will have a minimum value at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and this value is given by

$$U = x^2 + y^2 + w^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

Hence, the require value of utility = 1/3.

EXERCISE (F)

1. Find the equilibrium point from the following supply and demand equations

   (a) $p = \frac{2x}{35} + 5,\ p = \frac{3x}{35} + 10$, *respectively.*

   (b) $x = 20 - 3p - p^2,\ x = p - 1$, respectively. [Ans. $x = 35, p = 7$]

   [Ans. $x = 2, p = 3$]

2. The manufacturer's total cost function is

   $C = 500 + 25x + x^2$

   Find

   (a) Average cost function

   (b) Marginal cost function

   (c) Marginal cost when 10 units are produced

   (d) Actual cost of producing eleventh unit.

   $\left[\textbf{Ans.}\ (a)\frac{500}{x} + 25 + x\ (b)\,25 + 2x\ (c)\,45\ (d)\,46\right]$

3. Total cost is given by $C = 5000 + 1000q - 500q^2 + \frac{2}{3}q^3$

   (i) Find the MC function

   (ii) Find the expression for the slope of the MC curve.

   (iii) Find the average total cost function.

   (iv) At what value of q does MC equal AVC.

   $\left[\textbf{Ans.}\ (i)\,1000 - 1000q + 2q^2\ (ii) -1000 + 4q\ \ (iii)\frac{5000}{q} + 1000 - 5000q + \frac{2}{3}q^2\ (iv)\,q = 375\right]$

4. The total cost $C(x)$ of a firm is

   $C(x) = 0.25x^3 - 0.01x^2 - 80x + 1000$

   Where x is the output. Determine

   (a) The average cost.

   (b) The Marginal Average cost

   (c) The Marginal cost

   (d) The rate of change of MC w.r.t. x

   (e) The value of x for which MVC = AVC.

   $\left[\textbf{Ans.}\ (a)\,0.25x^2 - 0.01x - 80 + \frac{1000}{x}\ (b)\,0.5x - 0.01 - \frac{1000}{x}\ \ (c)\,0.75x^2 - 0.02x - 80\ (d)\,1.5x - 0.02\ (e)\,x = 1/50\right]$

5. The total cost function is $C = 2x^3 - 5x^2 + 7x$, find the marginal average cost function (MAC). What type of curve is the MAC? Find the points where it cut the axis.

6. The average cost function for a product is given by the equation. $AC = 0.04x^2 - 0.07x + \frac{500}{x}$, where x is the output, find the marginal cost function. What is the marginal cost when 100 units are produced. [Ans. $MC = 0.012x^2 - 0.14x$, for 100 units MC = 100]

7. If the total cost function is given as $C(x) = x + bx + cx^2$, where x is the quantity of output, show that $\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$. (Where MC and AC are marginal and average cost)

8. Given demand function, $p = a - bx$

Find

(a) Total revenue

(b) Average revenue

(c) Marginal revenue

[**Ans.** (a) $R = ax - bx^2$ (b) $AR = a - bx$ (c) $MR = a - 2bx$]

9. The total revenue received is a function of number of units sold (x) as.

$$R(x) = 500 + \frac{x^2}{10}$$

Find

(i) Average Revenue

(ii) Marginal Revenue

(iii) Marginal Revenue, when $x = 20$

(iv) Actual revenue form the sale of twenty first unit.

$\left[\textbf{Ans.}\ (i)\frac{500}{x}+\frac{x}{10}\ (ii)\frac{x}{5}\ (iii)\ 4\ (iv)\ 4.1\right]$

10. Show that demand curve $p = a(x + b)$ is downward sloping and convex from below (a and b are positive constants). Does the same property hold for marginal revenue curve?

11. A manufacturer has following cost and demand functions.

$$C(x) = 500 + 14x + \frac{x^2}{2}$$

And $$p = 500 - \frac{3x}{2}$$

Respectively. Find

(a) Marginal Cost

(b) Average Cost

(c) Marginal Average Cost.

(d) Show that $MAC = \frac{x(MC) - C(x)}{x^2}$

(e) For what value of x, AC is decreasing.

(f) When demand function increasing or decreasing.

$\left[\textbf{Ans.}\ (a)\ x+14, (b)\ \frac{500}{x}+14+\frac{x}{2}\ (c) -\frac{500}{x^2}+\frac{1}{2}\ (e)\ 0<x<10\sqrt{10}\right]$

12. The total cost function $C(x) = \sqrt{ax+b} + c$. Find an expression for marginal cost and show that it decreases as the output increases.

13. The rate of change of total cost (y) of a commodity per unit change of output (x) is called the marginal cost of a commodity. If there exist a relation between y and x in the form

$$y = 3x\left[\frac{x+7}{x+5}\right] + 5$$

Prove that marginal cost falls continuously as the output increases.

14. The cost function of a firm is given by $C(x) = 300x - 10x^2 + \frac{x^3}{3}$. Calculate the output at which the marginal cost is minimum. Will it be the same for the output at which average cost is minimum?

[**Ans.** $x = 15$]

15. The total revenue function of a firm is given by $R = 21q - q^2$, and its total cost function by $C = \frac{1}{3}q^3 - 3q^2 - 7q + 16$ where, q is the output.

Determine the output at which (i) the total revenue is maximum, and (ii) the total cost is minimum.

[**Ans.** 10.5, 7]

16. The unit demand function is $x = \frac{1}{3}(25 - 2p)$, where x is the number of units, and p is the price. If the average cost per unit be ₹ 40, then find,

(*i*) the revenue function R in terms of the price p.

(*ii*) the cost function C, (*iii*) the profit function P, (*iv*) the price per unit that maximizes the profit function, and (*v*) the maximum profit.

[**Ans.** (*i*) $\frac{1}{3}(25-2p)p$, (*ii*) $C(x) = 40 \cdot \frac{1}{3}(25-2p)$, (*iii*) $\frac{1}{3}[-2p^2 + 105p - 1000]$ (*iv*) $p = \frac{105}{4}$; ₹ 287.29]

17. The demand function of a firm is $p = 500 - 0.2x$, and its cost function is $C = 25x + 10{,}000$ (Where p = Price, x = output, and C = cost. Find the output at which the profit of the firm is maximum. Also, ascertain the price it will charge.

[**Ans.** 1187.5, 2625]

18. A monopolist firm has the following total cost, and demand functions respectively:
$C = ax^2 + bx + c$, $p = \beta - \alpha x$
Find the profit maximizing output level when the firm is assumed to fix (*i*) the output,
(*ii*) the price

[**Ans.** $x = \frac{\beta - b}{2(a+\alpha)}$ in both the cases]

19. A monopolist produces x units of a commodity at a total cost of $\eta = ax^2 + bx + c$, and the demand law for the same is given by $p = \alpha - \beta x$. A tax of ₹ K a unit is levied by the government. The manufacturer adds the tax to his cost.

(*i*) Find the price and the output both before and after the tax.

(*ii*) Show that the tax brings in the maximum return when $K = \frac{1}{2}(\alpha - b)$.

[**Ans.** $\frac{\alpha - b - K}{2(a+\beta)}$]

20. The production function of a firm is given by $Q = 8LK - L^2 - K^2$. Find the MPL and MPK, and show that $L.\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K} = 2Q$.

[**Ans.** 8K – 2L, 8L – 2K, Proved]

21. Give the production function, P = 4KL – 2K2 – L2. Find the maximum P with the constraint L + K = 10.

[**Ans.** 40/7]

22. Given the production function, $P = L^2 - 2KL + 2K^2$, where L stands for labour, and K for capital. Find the marginal product of labour while L = 2, and K = 3.

[**Ans.** 8]

23. Given, the Cobb-Douglas production function. $P = 10\, L^{1.25} K^{.5}$. Find the constant levels for (a) K is the fixed cost at 100, and L rises 5, 10, 15 (b) L is fixed at 100, and K rises 5, 10, 15, (c) When L is 10, and K is 15.

[**Ans.** (a) 747.67, 1778.28, 2951.98, (b) 707.11, 999.97, 1223.86, (c) 30...

24. The XYZ Co. Ltd. has the marginal revenue function by $MR = 20x = 2x^2$, and the marginal cost function by $MC = 81 - 16x + x^2$.

Find the profit maximizing output, and the total profit, at the optimal output level.

[**Ans.** $x = 9$, ...

25. The rate of change of the total cost (Y) of a commodity per unit change of output (X) is called the marginal cost of the commodity. If there exists a relation between Y and X in the function $Y = 3X\left(\frac{X+7}{X+5}\right) + 5$, then prove that the marginal cost falls continuously as the output increases.

15. MARGINAL ANALYSIS AND DIFFERENTIATION

By a marginal function we mean an equation that expresses the variation on the margin *i.e.* for a very small variation of x from a given value of x there is a small variation of y from a given value of y. Thus the marginal cost or revenue at a certain level of output is the change in the cost or revenue that results when the output is increased by a very small amount from the level. The process of Analysing a marginal function by using the method of maxima and minima is called marginal analysis.

Marginal Cost is the change in total cost associated with a change in output. The total cost is the sum of fixed cost and variable cost.

$$\text{Thus, the Marginal Cost} = \frac{\text{Change in total cost}}{\text{Change in output}} = \frac{\Delta_C}{\Delta_x} = \frac{dC}{dx}$$

Marginal cost is obtained by $MC = \frac{dC}{dx}$ and the cost minimization of a function can be find out by the method of finding maxima and minima.

Steps to Find out the Cost Minimisation :

Step–I : Find $\frac{dC}{dx}$ for the given cost function.

Step–II : Determine the values of x which makes $\frac{dC}{dx} = 0$

Step –III : Find $\frac{d^2C}{dx^2}$ and evaluate the same straight way or by putting $x = a$ in it, if x exist. If the above $\frac{d^2C}{dx^2}$ gives a +ve, the cost function is said to be at a **minimum at $x = a$** and the **marginal cost** is also minimum at a.

Similarly, **Marginal Revenue is** the addition made to total revenue by the sale of an additional unit of the product in the marginal.

We have, Total Revenue (R) = Price per unit × No. of out put

$$\therefore \text{Marginal Revenue (MC)} = \frac{\Delta R}{\Delta x} = \frac{dR}{dx}$$

The Marginal Revenue function is analysed by using the method of maxima and minima and if $\frac{d^2R}{dx^2}$ gives a –ve, the revenue function is said to be at a **maximum at $x = a$** and the marginal revenue is also maximum at a.

The excess of income from the sale of production over the costs of production is called **profit**.

Thus, Profit = Total Revenue – Total Cost

If total cost function is $C = f(x)$

Total Revenue function is $R = g(x)$

Total profit = $R - C = f(x) - g(x)$

Steps to Find out Profit Maximisation :

Step–I : Find $\frac{dP}{dx}$ for the function $P = R - C$

Step–II : Determine the values of x which make $\frac{dP}{dx} = 0$

Step–III : Find $\frac{d^2P}{dx^2}$ by putting $x = a$ in it.

If $\frac{d^2P}{dx^2}$ gives a –ve value, the profit function is said to be **maximum at $x = a$.**

By an average function, on the other hand, we mean an equation that expresses the variation of one quantity, say y over a specific range of values of another quantity say x *i.e.* from a given value to a certain selected value say from x to $x + A_x$ or say from 10 to 12 (where $x = 10$ and $A_x = 2$)

Thus, an average Cost (or revenue) is the ratio between the total cost (or revenue) and whole of the output concerned.

Thus, **Average Cost** is obtained by $AC = \frac{C}{x}$

And **Average Revenue** is obtained by $AR = \frac{R}{x}$

Where $C = f(x)$ and $R = g(x)$ and x = quantity of the product.

Further, by a total function we mean a function of the output (*i.e.* x)

Thus, the total cost is obtained by $C = f(x)$ = Cost per unit × quantity.

And, the total Revenue is obtained by $R = f(x)$ = Price per unit × quantity.

ILLUSTRATION 28. The average cost function (AC) for a commodity is given by

$$AC = x + 5 + \frac{36}{x}$$

in terms of output x. Find the total cost C and the marginal cost (MC) as function of x.

SOLUTION

Given, Average Cost (AC) $= x + 5 + \frac{36}{x}$ and output $= x$

Total Cost = Average Cost × No. of Output = $\left(x+5+\frac{36}{x}\right)\times x$

∴ $C = x^2+5x+36$

⇒ Now, Marginal Cost (MC) = $\frac{dC}{dx} = \frac{d}{dx}(x^2+5x+36) = 2x+5$

Thus, MC = 2x+5

Hence, the total cost function is $C = x^2+5x+36$ and the Marginal Cost function is $MC = 2x+5$.

ILLUSTRATION 29. The cost function of a firm be given by the following function :

$C = 300x - 10x^2 + \frac{1}{3}x^3$, where C stands for cost and x for output. Calculate :

(i) Output at which marginal cost is minimum

(ii) Output at which Average cost is minimum

(iii) Output at which Average Cost is equal to marginal Cost.

SOLUTION

(i) $C = 300x - 10x^2 + \frac{1}{3}x^3$

Marginal cost function, $\frac{d(C)}{dx} = \frac{d}{dx}\left(300x - 10x^2 + \frac{1}{3}x^3\right)$

Or $MC = 300 - 20x + x^2$

Analysing through the method of Maxima and minima we have

Step–I : $\frac{d}{dx}(MC) = \frac{d}{dx}(300 - 20x + x^2) = -20 + 2x$

Step–II : Considering $\frac{d(MC)}{dx} = 0$ or $-20 + 2x = 0,\ x = 10$

Step–III : $\frac{d^2(MC)}{dx^2} = \frac{d}{dx}\left(\frac{d(MC)}{dx}\right) = \frac{d}{dx}(-20+x^2) = 2x$

Putting the value of $x = 10$, $\frac{d^2(MC)}{dx^2} = 2\times 10 = 20$ *i.e.* > 0

Thus, the marginal Cost is minimum at $x = 10$.

(ii) Average Cost (AC) = $\frac{\text{Total cost}}{\text{No. of output}} = \frac{300x - 10x^3 + \frac{1}{3}x^3}{x}$

∴ $AC = 300 - 10x + \frac{1}{3}x^2$

For the analysis on minimization of cost, we have to follow the method of maxima and minima.

Step–I : $\frac{d}{dx}(AC) = \frac{d}{dx}\left(300 - 10x + \frac{1}{3}x^2\right) = -10 + \frac{2}{3}x$

Step–II : Considering $\frac{d(AC)}{dx} = 0$ or $-10 + \frac{2}{3}x = 0,\ x = 15$

Step–III : $\frac{d^2(AC)}{dx^2} = \frac{d}{dx}\left(\frac{d(AC)}{dx}\right) = \frac{d}{dx}\left(-10+\frac{2}{3}x\right) = \frac{2}{3}$ *i.e.* > 0

Thus, the average cost is minimum at $x = 15$

(iii) According to the Proposition,

Average cost function = Marginal Cost function.

$300 - 10x + \frac{1}{3}x^2 = 300 - 20x + x^2$

⇒ $\frac{2}{3}x^2 = 10x$ or $x = \frac{3}{2}\times 10 = 15$

⇒ Hence, for output 15, the average cost is equal to the marginal cost

ILLUSTRATION 30. Find the profit maximizing output from the following Revenue and cost functions :

$R(x) = 1{,}000x - 2x^2$

$C(x) = x^3 - 59x^2 + 1315x + 2000$

SOLUTION

We have, Profit (P) = Revenue – Cost = R(x) – C(x)

Or $P = (1000x - 2x^2) - (x^3 - 59x^2 + 1315x + 2000)$

∴ $P = -x^3 + 57x^2 - 315x - 2000$

By method of maxima and minima we have,

Step–1 : $\frac{dP}{dx} = \frac{d}{dx}(-x^3 + 57x^2 - 315x - 2000) = -3x^2 + 114x - 315$

Step–2 : Considering $\frac{dP}{dx} = 0,\ -3x^2 + 114x - 315 = 0,\ x = 3$ or 35

Step–3 : $\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(-3x^2 + 114x - 315) = -6x + 114$

Putting the value of $x = 3$, $\frac{d^2P}{dx^2} = -6\times 3 + 114 = 96$ *i.e.* > 0

Putting the value of $x = 35$, $\frac{d^2P}{dx^2} = -6\times 35 + 114 = -106$ *i.e.* < 0 and the profit is maximum at $x = 35$.

Hence the profit maximizing output is 35.

ILLUSTRATION 31. The unit demand function is $x = \frac{1}{3}(25 - 2p)$ where x is the number of units and p is the price. Let the average Cost per unit be ₹ 40. Find,

(a) the revenue function R in terms of price P,

(b) the cost function C

(c) the profit function P,

(d) the price per unit that maximizes the profit function,

(e) the maximum profit

SOLUTION

(a) Revenue $=$ Price per unit $\times$ Quantity demanded

Revenue Function or $R(x) = p.\,x = p \times \frac{1}{3}(25-2p) = \frac{1}{3}(25p - 2p^2)$

(b) Cost $=$ Cost per unit $\times$ Quantity

Cost function or $C(x) = 40 \times \frac{1}{3}(25-2p) = \frac{40}{3}(25-2p)$

(c) Profit (P) $=$ Revenue $-$ Cost $= R(x) - C(x)$

$$P = \frac{1}{3}(25p - 2p^2) - \frac{40}{3}(25-2p)$$

Or
$$P = \frac{1}{3}[25p - 2p^2 - 1000 + 80p]$$

Profit function or
$$P = \frac{1}{3}[-2p^2 + 105p - 1000]$$

(d) In order to find the price per unit that maximizes the profit function, we have to follow the method of maxima and minima.

Step–I :
$$\frac{dP}{dp} = \frac{d}{dp}\left(\frac{1}{3}[-2p^2 + 105p - 1000]\right)$$
$$= \frac{1}{3}[-4p + 105] = -\frac{4}{3}p + 35$$

Step–II: Considering $\frac{dP}{dp} = 0$, $-\frac{4}{3}p + 35 = 0$ or $p = \frac{105}{4} = 26.25$

Step–III :
$$\frac{d^2P}{dp^2} = \frac{d}{dp}\left(\frac{dP}{dp}\right) = \frac{d}{dp}\left(-\frac{4}{3}p + 45\right) = -4/3 \text{ i.e. } < 0$$

Profit is maximum at price $p = 26.25$

(f) Maximum Profit $= \frac{1}{3}[-2p^2 + 105p - 1000]$
$$= \frac{1}{3}\left[\left(-2 \times \frac{105}{4} \times \frac{105}{4}\right) - \left(105 \times \frac{105}{4}\right) - 1{,}000\right]$$
$$= ₹\,126.04$$

ILLUSTRATION 32. The cost function $C(x)$ for producing x units of a commodity is given by $C(x) = \frac{1}{3}x^3 - 5x^2 + 75x + 10$. At what level of output the marginal cost (*i.e.* $\frac{dC}{dx}$) attains its minimum? What is the marginal cost of this level of production ?

SOLUTION

The cost function $C(x)$ for producing x units is given by.

$$C(x) = \frac{1}{3}x^3 - 5x^2 + 75x + 10$$

The marginal cost (y) or $\frac{dC}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^3 - 5x^2 + 75x + 10\right) = \frac{1}{3} \times 3 \times x^{3-1} - 5 \times 2 \times x^{2-1} + 75$

$$y = x^2 - 10x + 75$$

or $\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 10x + 75) = 2x - 10$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x - 10) = 2 \text{ i.e. } > 0$$

For marginal cost 'y' to be minimum we must have $\frac{dy}{dx} = 0$

i.e.
$$2x - 10 = 0 \text{ or } x = 5$$

At $x = 5$,
$$\frac{d^2y}{dx^2} = 2 \text{ i.e. } > 0 \text{ or } +\text{ve}$$

Hence, the marginal cost 'y' is minimum at output level 5 units and the minimum marginal cost at this level is.

$$y = x^2 - 10x + 75 = 5^2 - 10 \times 5 + 75 = 50 \text{ or } ₹\,50$$

ILLUSTRATION 33. The total cost function of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$, where C is the total cost and x is the output. A tax at the rate of ₹ 2 per unit of output is imposed and the producer adds it to his cost. If the demand function is given by $p = 2{,}530 - 5x$, where ₹ p is the Price per unit of output, find the profit maximising output and the price at the level.

SOLUTION

The total cost function $C(x)$ of x units

$$C(x) = \text{Total Cost of firm} + \text{taxes}$$
$$= \left(\frac{1}{3}x^3 - 5x^2 + 28x + 10\right) + 2x = \frac{1}{3}x^3 - 5x^2 + 30x + 10$$

If $R(x)$ is the total Revenue, $R(x) = p.x = (2{,}530 - 5x)\,x = 2{,}530\,x - 5x^2$

If $P(x)$ is the Profit on sale of x units, then

$$P(x) = R(x) - C(x) = (2{,}530\,x - 5x^2) - \left(\frac{1}{3}x^3 - 5x^2 + 30x + 10\right)$$

$\Rightarrow$
$$P(x) = 2500\,x - \frac{1}{3}x^3 - 10$$

Analysing by the method of maxima and minima we have.

Step–I : $\frac{d}{dx}(P) = \frac{d}{dx}\left(2{,}500x - \frac{1}{3}x^3 - 10\right) = 2500 - x^2$

Step–II : Determine the value of x which makes $\left(\frac{dP}{dx}\right) = 0$.

Thus, $2{,}500 - x^2 = 0$ or $x = 50$

Step–III : $\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(2{,}500 - x^2) = -2x$

At $x = 50$, $\frac{d^2P}{dx^2} = -2x = 2 \times 50 = -100$ *i.e.* < 0 and the profit is maximum at $x = 50$.

Hence, the profit maximizing output is 50 and the price p at $x = 50$ is given by,

$$p = 2{,}530 - 5x = 2{,}530 - 5\times50 = ₹\ 2{,}280$$

EXERCISE (G)

1. The cost C of manufacturing a certain article is given by the formula: $C = 5 + \frac{48}{x} + 3x^2$, where x is the number of articles manufactured. Find the minimum value of C. (Ans. 4

2. A firm produces an output of x tones of a certain product at a total cost given by C = ₹ $(x^3 - 4x^2 + 7x)$. Find at what level of output average cost is minimum and what level will it be? (Ans. 2 tonnes, ₹ 3

3. Find the maximum and minimum values of $f(x) = 1/2x^4 - x^2 + 1$. If the total cost (C) and total revenue (R) for a company are given by $C = 20 + 4x$, $R = 30x - x^2$, where x is the output, find the output at which the profit is maximum. What is the maximum profit. (Ans. Mx – 1 & Min – ½

4. A manufacture can sell z items $(z > 0)$ at a price of ₹ $(330 - x)$ each. The cost of producing items is ₹ $(x^2 + 10x + 12)$. How many items should he sell to make the maximum profit? Determine also the maximum profit. (Ans. 80 units

5. The efficiency E of a small manufacturing concern depends on the number of workers W and is given by $10E = \frac{W^3}{40} + 30W - 392$. Find the strength of the workers which gives maximum efficiency. (Ans. 20

[**Hints.** $10\frac{dE}{dW} = \frac{3}{40}W^2 + and\ 10\frac{d^2E}{dW^2} = -\frac{3}{20}W, \frac{dE}{dW} = 0$ gives 3W2 = 1200, or W = +20.

But W cannot be negative. W = 20 and at W = 20, $\frac{d^2E}{dW^2} = -\frac{3}{10} < 0$, etc.]

6. A steel plant producers x tones of steel per week at a total cost of ₹ $(1/2x^3 - 7x^2 + 111x + 50)$. Find the output level at which the marginal cost attains its minimum. (Use other concept of sign of derivative as used in finding extreme values.) (Ans. 7 tonnes

7. A firm produces x tones of output at a total cost C = ₹ $1/10x^3 - 5x^2 + 10x + 5$).

At what level of output will the marginal and the average variable cost attain their respective minima? (Ans. 50/3 tonnes, 25 tonnes

8. The demand function for a particular commodity is $y = 15$ for $0 \le x \le 8$, where y is the price per unit and x is the number of units demanded. Determine the price and the quantity for which revenue is maximum. (Ans. 15/2, 3 units

[**Hints.** If R be the total Revenue, then $R = I\ xy\ I = 15xe^{-x/3}$

$$\frac{dR}{dx} = 15\left[1.e^{\frac{-x}{3}} + x.e^{\frac{-x}{3}}\left(-\frac{1}{3}\right)\right] = 15e^{-\frac{x}{3}}\left(1 - \frac{x}{3}\right)$$

$$\therefore \text{ and } \frac{d^2R}{dx^2} = 15\times\frac{1}{3}e^{\frac{-x}{3}}\left(1 - \frac{x}{3}\right) + 15e^{\frac{-x}{3}} \times -\frac{1}{3} = -5e^{-\frac{x}{3}}\left(2 - \frac{x}{3}\right)$$

Now apply second derivative test for maxima]

9. The total cost function of a firm is $C = 1/3x^3 - 4x^2 + 22x + 9$ where C is the total cost and x is the output. A tax at the rate of ₹ 3 per unit of output is imposed and the producer adds it to his cost. If the demand function is given yb $p = 1625 - 4x$, where ₹ p is the price per unit of output, find the profit maximizing output and the price at the level. (Ans. 40 units, ₹ 1465)

10. A radio manufacturer finds that he can sell x radios per week at ₹ p each, where $p = 2(100 - \frac{z}{4})$. His cost of production of z radios per week is ₹ $(120x + \frac{x^2}{2})$. Show that his profit is maximum when the production is 40 radios per week. Find also his maximum profit per week. (Ans. 1600)

[**Hints.** R = total revenue = ₹ px = ₹ $2(100 - \frac{z}{4})x$ = ₹ $(200x - \frac{x^2}{2})$

and Total cost (TC) = ₹ $(12x + 2\frac{x^2}{2})$]

$\therefore$ P = total profit = R – TC = ₹ $(200x - \frac{x^2}{2})$ – ₹ $(120x + \frac{x^2}{2})$ = ₹ $(80x - x^2)$.

$\therefore \frac{dP}{dx} = 80 - 2x$ and $\frac{d^2P}{dx^2} = -2$, etc.]

11. $p = \frac{150}{q^2 + 2} - 4$ represents the demand function for a product where q is the price per unit units. Determine the marginal revenue function. (Ans. $150(2-q^2/(q^2+2)^{2-4})$

[**Hints.** If R be the total revenue for q units, then $R = pq = \frac{150}{q^2 + 2} - 4q$.

$\therefore$ Marginal Revenue = $\frac{dR}{dq}$ = etc.]

12. (i) A manufacturer can sell x items per month at a price $p = (100 - 2x)$ Rupees. Manufacturer's cost of production, ₹ y, of x items is given by $y = 2x + 1000$. Find the number of items to be produced per month to yield the maximum profit. (Ans. 75)

(ii) The price p per unit at which a company can sell all that it produces is given by the function $P(x) = 300 - 4x$. the cost function is $C(x) = 500 + 28x$ where x is the number of units produced. Find x so that the profit is maximum. (Ans. 34)

[**Hints.** P(x) = Profit function = Revenue –Cost = $px - C(x)$ $300x - 4x^2 - (500 + 28x)$ = etc.]

13. A firm produces x tones of output at a total cost C = ₹ $\left(\frac{1}{10}x^3 - 5x^2 + 10x + 5\right)$. At what level of output will the marginal cost and the average variable cost attain their respective minima?

(**Ans.** 50/3 tonnes)

14. The total cost C of output q is given by $C = 300q - 10q^2 + 1/3q^3$. find the output levels at which the marginal cost and the average cost attain their respective minima. (**Ans.** 10 units, 15 units)

15. If n be the number of workers employed, the average cost of production is given by $C = 24n + \frac{3}{2(n-4)}$. Show that $n = 4\frac{1}{4}$ will make C minimum. Will you then advice to employ 4 to 5 workers?

(**Ans.** 5 worked or 4 full time)

[**Hints.** $\frac{dC}{dn} = 24 - \frac{3}{2(n-4)^2}$ and $\frac{d^2C}{dn^2} = \frac{3}{(n-4)^3}$

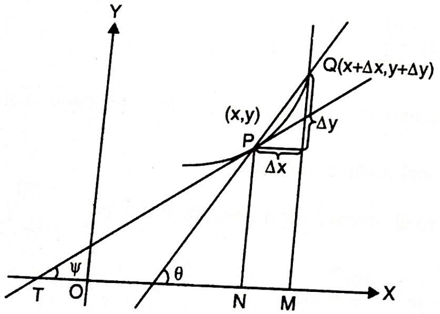
$\therefore$ C will be minimum when $\frac{dC}{dn} = 0$ and $\frac{d^2C}{dn^2} > 0$. $\frac{dC}{dn} = 0$ given $n = 4\frac{1}{4}$ or $3\frac{3}{4}$.

$\therefore$ At $C = 4\frac{1}{4}, \frac{d^2C}{dn^2} = 3/\left(\frac{1}{4}\right)^3 > 0$, but at $C = 3\frac{3}{4}, \frac{d^2C}{dn^2} < 0$.

16. DERIVATIVE AS THE SLOPE OF A TANGENT TO THE CURVE y = f(x)

(Geometrical Interpretation of the derivative at a point)

Let P(x, y) be any given point on the curve $y = f(x)$ with reference to the rectangular axes OX and OY, and Q be a neighbouring point $(x + \Delta x, y + \Delta y)$ on either side of P. Supposes $y = f(x)$ is continuous at P. Let PN, QM be perpendiculars to OX and PL perpendicular to QM. Then PL = NM = OM – ON = $x + \Delta x - x = \Delta x$ and LQ = MQ – ML = MQ – NP = $y + \Delta y - y = \Delta y$.



The slope of the chord PQ

$$= \frac{LQ}{PL} = \frac{\Delta y}{\Delta x} = \frac{\text{increment of } y}{\text{increment of } x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad ...(1)$$

Let, Q tend to P along the curve; then $\Delta x \to 0$ and the chord PQ may tend to a definite limiting position PT (which is called the tangent to the curve at P) as Q→ P from either side and

$$\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} \text{ or, } f'(x)$$

Thus, in the limit when $\Delta x \to 0$, from (1) we have the slop of the tangent PT at P = $\frac{dy}{dx}$. Hence geometrically $\frac{dy}{dx}$ represents the slope (or gradient) of the tangent line to the curve $y = f(x)$ at the point (x, y).

[If ψ (psi) be the inclination of the tangent line with the positive direction of the x-axis, then the slope of the line = tan ψ, *i.e.*, $\frac{dy}{dx} = \tan\psi$.]

If $\frac{dy}{dx} = 0$. then slop = 0. This implies that the tangent line at (x, y) is parallel to the x-axis. If $\frac{dy}{dx} = 0$, then the tangent line at (x, y) is parallel to the y –axis.

Remarks 1. From Plane Analytic Geometry, the equation of the tangent to the curve $y = f(x)$ at any point P(x, y) is.

$Y - y = \frac{dy}{dx}(X - x)$, where $\frac{dy}{dx}$ = slope of the tangent at P, and the equation of the normal at P(x, y) is

$Y - y = \frac{-1}{\frac{dy}{dx}}(X - x)$, where $\frac{dy}{dx} \neq 0$ at P, or, $(X - x) + (Y - y)\frac{dy}{dx} = 0$.

Thus, the equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) may be written as $y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}(x - x_1)$, where the slope of the tangent at (x_1, y_1) is $\left[\frac{dy}{dx}\right]_{(x_1, y_1)}$.

The two tangents to the curve $y = f(x)$ will be (*i*) parallel if their slopes are equal and (ii) perpendicular to each other if the product of their slopes is –1.

Remarks 2 : If $\frac{dy}{dx}$ is positive at P(x, y) *i.e.*, if $f'(x) > 0$. then the tangent line at P(x, y) makes an acute angle with the positive direction of the x-axis. But if $\frac{dy}{dx}$ is negative at Q(x, y), *i.e.* if $f'(x) < 0$, then the tangent line at Q makes an obtuse angle with the positive direction of the x-axis.

If $\frac{dy}{dx} = 0$ at (x, y) *i.e.*, $f'(x) = 0$, then the tangent line is parallel to the x-axis. From the figure, we see that at A or B the tangent line is parallel to the x-axis, *i.e.*, at A or B, $\frac{dy}{dx} = 0$.

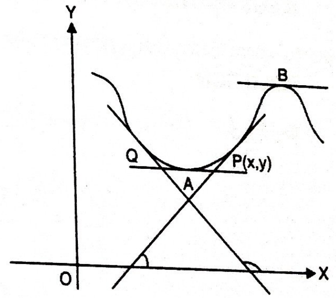


ILLUSTRATION 34. Find the slope of the curve $y = x^2 - 3x + 4$ when $x = 1$ and hence find the equation of the tangent at the point (1, 2).

SOLUTION

The equation of the curve is $y = x^2 - 3x + 4$

$\therefore \frac{dy}{dx} = 2x - 3$. At $x = 1$, $\frac{dy}{dx} = (2\times1) - 3 = -1$.

Hence the slope of the curve (1) when $x = 1$ is -1.

The required equation of the tangent to the curve (1) at the point (1, 2) is

$$y-2 = \left[\frac{dy}{dx}\right]_{(1,\,2)}(x-1)\text{, or } y-2 = -1(x-1)$$

or, $$y-2 = -x+1 \text{ or } x+y=3.$$

ILLUSTRATION 35. Find the slope of the curve $x^3 - 2x^2y^2 + 5x + y = 5$ at the point (1, 1).

SOLUTION

The curve is $x^3 - 2x^2y^2 + 5x + y = 5$

Differentiating both sides with respect to 'x', we get

$$3x^2 - 2\left(2x.y^2 + x^2.2y\frac{dy}{dx}\right) + 5.1 + \frac{dy}{dx} = 0,$$

or, $$3x^2 - 4xy^2 - 4x^2y\frac{dy}{dx} + 5 + \frac{dy}{dx} = 0,$$

or $$\frac{dy}{dx}(1-4x^2y) = 4xy^2 - 3x^2 - 5,$$

or, $$\frac{dy}{dx} = \frac{4xy^2 - 3x^2 - 5}{1-4x^2y}$$

At the point (1, 1), $$\frac{dy}{dx} = \frac{4.1.1^2 - 3.1^2 - 3.1^2 5}{1-4.1^2.1} = \frac{4-3-5}{1-4} = \frac{-4}{-3} = \frac{4}{3}$$

ILLUSTRATION 36. Find the slope of the curve $y = \frac{x^2-12}{x-4}$ at the point (0, 3) and determine the points where the tangent is parallel to the axis of x.

SOLUTION

We have $$y = \frac{x^2-12}{x-4}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x-4).\frac{d}{dx}(x^2-12) - (x^2-12).\frac{d}{dx}(x-4)}{(x-4)^2}$$

$$= \frac{(x-4).2x - (x^2-12).1}{(x-4)^2} = \frac{x^2-8x+12}{(x-4)^2}$$

At the point (0, 3), $x = 0$ and hence

$$\frac{dy}{dx} = \frac{0^2 - 8.0 + 12}{(0-4)^2} = \frac{12}{16} = \frac{3}{4}$$

Hence, the slope of the curve at the point (0, 3) is $\frac{3}{4}$.

The points at which the tangents are parallel to the axis of x is given by

$$\frac{dy}{dx} = 0, \text{ i.e. } x^2 - 8x + 12 = 0$$

or. $$(x-2)(x-6) = 0;\ \therefore x = 2, 6.$$

If $x = 2$, from (1), $y = \frac{2^2-12}{2-4} = \frac{-8}{-2} = 4$ and if $x = 6$, $y = \frac{6^2-12}{6-4} = \frac{24}{2} = 12$.

Hence the points at which the tangents are parallel to the axis of x are (2, 4) and (6, 12).

ILLUSTRATION 37. Determine the coefficients A and B so that the curve $y = Ax^2 + 3x + B$ may pass through the point (0, 1) and have is tangent parallel to the axis of x at the point for which $x = 0.75$.

SOLUTION

We have $$y = Ax^2 + 3x + B. \quad \ldots(1)$$

Since the curve (1) passes through the point (0, 1), we have $1 = B$, i.e., $B = 1$.

Differentiating both sides of (1) w.r.t. x, we get

$$\frac{dy}{dx} = 2Ax + 3.$$

At $x = 0.75$, the value of $\frac{dy}{dx} = 2A \times 0.75 + 3 = 1.5A + 3$.

Since the tangent at the point $x = 0.75$ is parallel to the axis of x, we have $1.5A + 3 = 0$

$$\left[\because \frac{dy}{dx} \text{ at } x = 0.75 \text{ is } 0.\right]$$

or, $$1.5A = -3\text{, or, } A = -\frac{3}{1.5} = -2.$$

Hence, $$A = -2 \text{ and } B = 1.$$

ILLUSTRATION 38. Find the slope of the curve $x = y^2 - 4y$ at the points where it crosses the y-axis.

SOLUTION

For the points of intersection of the curve with the y-axis we have $x = 0$ and $x = y^2 - 4y$; $\therefore y^2 - 4y = 0$.

or, $$y(y-4) = 0;\ \therefore y = 0, 4$$

$\therefore$ The points of intersection of the curve and the y-axis are (0, 0) and (0,4).

We have $$\frac{dx}{dy} = 2y - 4 = 2(y-2).$$

$$\therefore \quad \frac{dy}{dx} = 1/\frac{dx}{dy} = \frac{1}{2(y-2)}$$

At $$(0, 0), \frac{dy}{dx} = \frac{1}{2(0-2)} = -\frac{1}{4}.$$

And at $$(0, 4), \frac{dy}{dx} = \frac{1}{2(4-2)} = \frac{1}{4}$$

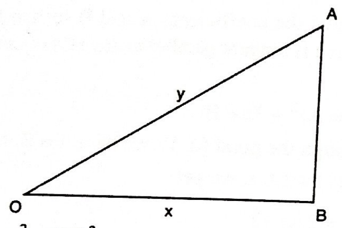
Hence the required slopes are $-\frac{1}{4}$ and $\frac{1}{4}$.

ILLUSTRATION 39. A man is walking at the rate of 5 km. per hour towards the foot of a tower of 30 meters high. At what rate is he approaching the top of the tower when he is 40 metres from the foot of the tower ?

SOLUTION

Let AB be the tower of height 30 metres, B being its foot, and let O be the position of the man at any time t such that OB = x meters. Let OA = y , the y is the distance of the man from the top of the tower at time t.

From, the right angled ΔAOB, we have



$$y^2 = x^2 + (30)^2 \quad ...(1)$$

Differentiating both sides with respect to 't', we get

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + 0, \text{ or, } \frac{dy}{dt} = \frac{x}{y} \times \frac{dx}{dt} \quad ...(2)$$

When $x = 40$ metres, from (1), $y^2 = (40)^2 + (30)^2 = 2500$; $\therefore$ $y = 50$ metres.

Since the man is walking at the rate of 5 km. per hour;

$$\therefore \quad \frac{dy}{dt} = -5 \text{ km./hr.} \qquad [\because x \text{ decrease as } t \text{ increases}]$$

$\therefore$ From (2), when $x = 40$ m, we get

$$\frac{dy}{dt} = \frac{40}{50} \times -5 = -4 \text{ km./hr.}$$

[Negative sign indicates that y decreases when t increases.]

Hence, the man is approaching the top at the rate of 4 km./hr.

EXERCISE (H)

1. A man is walking at the rate of km. per hour towards the foot of a tower 60 meters high. A what rate is he approaching the top when he is 8-0 metres from the foot of the tower.
2. An inverted cone has height of 15 cms. And a base of radius 7.5 cms. Water is poured into it at the rate of 2.2 c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 7 cms.
3. Find the value of $\sqrt{6.33}$, given $\sqrt{6.25} = 2.5$ by the method of differential.

   [Hints. $f(x + \Delta x) = f(x) + f'(x)\,dx$; here $x = 6.25$, $\Delta x = 0.08$, $f(x) = \sqrt{x}$, $f'(x) = 1/2\sqrt{x}$ /]
4. (i) Find the slope of the tangent line at the point (0, 5) of the curve $y = \frac{1}{5}(x^2 + 10x - 15)$. At what point of the curve the slope of the tangent line is 8/5 ?

   (ii) Find the gradient of the curve $y = 2x^3 - 3x^2 - 12x + 8$ when $x = 0$.

5. Find the slope of the tangent to the curve $y = \frac{x^2 - 15}{x - 4}$ at the point (0, 2) and determine the points where the tangent is parallel to the axis of x.
6. Determine the coefficients a and b so that the curve $y = ax^2 - 6x + b$ may pass through the point (0, 2) and have its tangent parallel to the axis of x at the point for which $x = 1.5$.

17. CONCAVITY AND CONVEXITY : POINT OF INFLEXION

Let $P(x,y)$ be any point on the curve $y = f(x)$. Then the slope of the tangent line at P varies with $P(x,y)$.

When the tangent line PT is below the curve, the Arc (or curve) is concave upward or convex downward (see fig (1)); but when the curve is below the tangent line PT, the Arc (or curve) is concave downward or convex upward (see fig (2)).

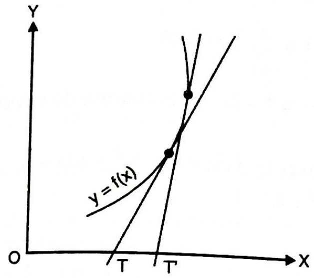


Fig. (1)

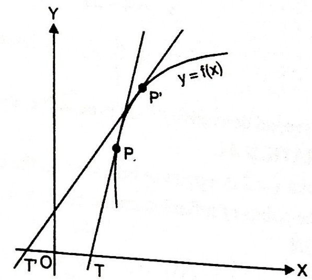


Fig. (2)

In fig (1) slope increases as x increases i.e., $f'(x)$ is an increasing function of x. Hence $f''(x)$ is positive. In fig (2) the slope of the tangent decreases as x increases i.e., $f'(x)$ is a decreasing function of x. Hence $f''(x)$ is negative.

Hence the curve $y = f(x)$ will be (*i*) concave upward (or convex downward) if the second derivative $f''(x)$ or $\frac{d^2y}{dx^2}$ is positive, and concave downward (or conved upward) if $f''(x)$ or $\frac{d^2y}{dx^2}$ is negative.

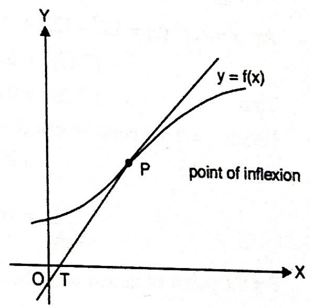


Fig. (3)

A point P of a curve $y = f(x)$ will be called a **point of inflexion** if the line at P crosses the curve i.e., if P separates arcs having opposite directions of bending (see fig (3). This means that the curve on one side of P is concave upward and on the other side it is concave downward or vice versa.

On the two sides (left and right sides) of a point of inflexion P, the second derivative $f''(x)$ will have opposite signs, and hence $f''(x) = 0$ at P, provided $f'''(x)$ is continuous at P.

The condition for a point P to be a point of inflexion on the curve $y = f(x)$ is that

At P, $f''(x)$ or $\frac{d^2y}{dx^2} = 0$ and $f'''(x)$ or $\frac{d^3y}{dy^3} \neq 0$.

ILLUSTRATION 40. Show that (i) $y = x^2$ is concave upward, and (ii) $y = 4 - 2x - x^2$ is concave downward.

SOLUTION

(i) $y = x^2$ i.e., $f(x) = x^2$

$$f'(x) \text{ or } \frac{dy}{dx} = 2x \text{ and } f''(x) \text{ or } \frac{d^2y}{dx^2} = 2 > 0.$$

Since the second derivation $f''(x)$ is positive, the curve $y = x^2$ is concave upward.

(ii) $y = 4 - 2x - x^2$, i.e., $f''(x) = 4 - 2x - x^2$.

$$f'(x) \text{ or } \frac{dy}{dx} = -2 - 2x \text{ and } f''(x) \text{ or } \frac{d^2y}{dx^2} = -2 < 0.$$

Since the second derivative $f''(x)$ is negative, the curve $y = 4 - 2x - x^2$ is concave downward.

ILLUSTRATION 41.

(i) Show that $x = 2$ is a point of inflexion on the curve given by $f(x) = x^3 - 6x^2 + 12x - 5$.

(ii) Find the points of inflexion on the curve $y = x^4 - 6x^2 + 8x - 1$.

SOLUTION

(i) We have $f(x) = x^3 - 6x^2 + 12x - 5$.

$$\therefore \frac{dy}{dx} \text{ or } f'(x) = 3x^2 - 12x + 12,\ \frac{d^2y}{dx^2} \text{ or } f''(x) = 6x - 12 \text{ and } \frac{d^3y}{dx^3} \text{ or } f'''(x) = 6.$$

At $x = 2, f'(x) = 3.2^2 - 12.2 + 12 = 12 - 24 + 12 = 0$

$f''(2) = 6.2 - 12 = 0$ and $f'''(2) = 6 \neq 0$.

Thus $f''(2) = 0$ and $f'''(2) \neq 0$.

Hence, $x = 2$ is a point of inflexion on the curve (1).

(ii) We have $y = x^4 - 6x^2 + 8x - 1$

$$\therefore \frac{dy}{dx} = 4x^3 - 12x + 8,\ \frac{d^2y}{dx^2} = 12x^2 - 12 \text{ and } \frac{d^3y}{dx^3} = 24x.$$

For the points of inflexion on the curve (1), we have

$$\frac{d^2y}{dx^2} = 0, \text{ or, } 12x^2 - 12 = 0, \text{ or, } x^2 = 1;\ \therefore x = \pm 1.$$

At $x = 1$, $\frac{d^3y}{dx^3} = 24.1 = 24 \neq 0$ and at $x = -1$; $\frac{d^3y}{dx^3} = 24 \times -1 = 24 \neq 0$.

If $x = 1$, then from (1), $y = 1 - 6.1 + 8.1 - 1 = 2$,

and if $x = -1$, then $y = (-1)^4 - 6.(-1)^2 + 8.(-1) = 1 - 6 - 8 - 1 = -14$

Thus at the point (1,2) and at the point (–1,–14), we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0.$$

Hence (1,2) and (–1, –14) are required points of inflexion on the curve (1).

ILLUSTRATION 42. Show that the curve $y = a^2(3 - x)$ has a point pf inflexion at the point (1,2).

SOLUTION

We have $y = x^2(3 - x) = 3x^2 - x^3$ (1)

$$\therefore \frac{dy}{dx} = 6x - 3x^2,\ \frac{d^2y}{dx^2} = 6 - 6x \text{ and } \frac{d^3y}{dx^3} = -6.$$

For the points of inflexion on the curve (1), we have

$$\frac{d^2y}{dx^2} = 0, \text{ or, } 6 - 6x = 0, \text{ or, } x = 1.$$

At $x = 1$, $\frac{d^3y}{dx^3} = -6 \neq 0$ i.e., $\left(\frac{d^3y}{dx^3}\right)_{x=1} \neq 0$.

Also if $x = 1$, from (1), $y = 3.1^2 - 1^3 = 2$.

Thus at the (1,2), we have $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$.

Hence the curve (1) has a point of inflexion at the point (1,2).

ILLUSTRATION 43. If the sides of a rectangle be x and y, express the perimeter P and the area A in terms of x and y. Prove (by the method of Calculus) that among all the rectangles with fixed perimeter P, the square is the one with maximum area.

SOLUTION

Perimeter of the rectangle = $2(x + y)$ units,

and area of the rectangle = Length × breadth = xy sq. units;

$\therefore$ $P = 2(x + y)$, ...(1)

and $A = xy$, ...(2)

where P is fixed.

From (1), $x + y = \frac{P}{2}$, or, $y = \frac{P}{2} - x$

$\therefore$ From (2), $A = xy = x\left(\frac{P}{2} - x\right) = \frac{Px}{2} - x^2$; $\therefore \frac{dA}{dx} = \frac{P}{2} - 2x$. [$\because$ P is fixed]

For the maximum value of A,

$$\frac{dA}{dx} = 0, \text{ or, } \frac{P}{2} - 2x = 0, \text{ or, } x = \frac{P}{4};\ \therefore y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = \frac{d^3y}{dx^3}\frac{d}{dx}\left(\frac{P}{2}\right) - \frac{d}{dx}(2x) = -2 < 0.$$

$\therefore$ A is maximum when $x = \frac{P}{4}$ and $y = \frac{P}{4}$ i.e., when $x = y$.

Hence among all the rectangles with fixed perimeter, the square is the one with maximum area.

ILLUSTRATION 44. A watermelon grower wishes to send his product to the city market as early as possible in the season to catch the prices. He can send now 6 tons at a profit of ₹ 60 per ton. By waiting he estimates that he can add 3 tons per week to what he can send now, but then the profit will be reduced by ₹5 per ton per week. How long should he wait for the maximum profit ? (Use the method of Calculus.)

SOLUTION

Let the watermelon grower wait x weeks for maximum profit. Then the quantity of products that can be sent to the city market = $(6 + 3x)$ tons and profit per ton = ₹ $(60 - 5x)$.

If $f(x)$ be the total profit in rupees, then

$$f(x) = (6 + 3x)(60 - 5x) = 360 + 150x - 15x^2.$$

$$\therefore \quad f'(x) = 150 - 30x = 30(5 - x) \text{ and } f''(x) = -30.$$

For the total profit $f(x)$ to be maximum, we have $f'(x) = 0$, i.e., $30(5 - x) = 0$, or, $x = 5$.

At $\quad x = 5, f''(5) = -30 < 0.$

Hence the watermelon grower should wait for 5 weeks for the maximum profit.

EXERCISE (I)

1. Find the point of inflexion, at which the tangent is parallel to the base of any of the following.
   (a) $y = x^4 - 5x^3$ (b) $y = 5x^3 - 7x^2$
   (c) $y = x^4 - 5x + 12$ (d) $y = e^{-x^2/2}$
2. Show the (i) $y = x^2 + 2x - 3$ is concave upward and (ii) $y = 5 - 3x - x^2$ is concave downward.
3. Show that (i) $x = 1$ is a point of inflexion on the curve $y = x^3 - 3x^2 + 3x - 5$, (ii) $x = 2$ is a point of inflexion to the curve $f(x) = x^4 - 4x^2 + 2x + 3$.
4. Show they $y = x^3 - 8$ has neither a maximum nor a minimum value. Does the curve have a point of inflexion?
5. Show that the function $f(x) = x^2 + \dfrac{250}{x}$ has a minimum value at $x = 5$.
6. Find the maximum and minimum values of the function
   (i) $x^3 - 3x + 1$ (ii) $2x^3 + 3x^2 - 36x + 10$, (iii) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$.
7. Investigate the following function for maximum and minimum:
   $$y = \frac{1}{3}x^3 - 2x^2 + 3x + 1.$$
8. Prove that the curve given by $3y = x^3 - 3x^2 - 9x + 11$ has a maximum at $x = -1$, minimum at $x = 3$ and a point of inflexion at $x = 1$
9. Find the maximum and minimum values of the function $x^5 - 5x^4 + 5x^3 - 1$. Discuss its nature at $x = 0$.
10. (i) Find the points of inflexion on the curve $y = x^4 - 6x^2 + 8x - 1$.
    (ii) Show that the curve $y = x^2(3 - x)$ has a point of inflexion at the point (1,2)

11. $p = \dfrac{150}{q^2 + 2} - 4$, represents the demand function for a product where p is the price per unit for q units. Determine the marginal revenue function.

    [**HINTS :** If R be the total revenue for q units, then $R = pq = \dfrac{150q}{q^2 + 2} - 4q$.

    $\therefore$ Marginal Revenue = $\dfrac{dR}{dq}$ = etc.]

12. (i) A manufacturer can sell x items per month at a price $p = (300 - 2x)$ rupees. Manufacturer's cost of production, ₹ y, of x items is given by $y = 2x + 1000$. Find the number of items to be produced per month to yield the maximum profit.

    (ii) The price 'p' per unit at which a company can sell all that it produces is given by the function $P(x) = 300 - 4x$. The cost function is $C(x) = 500 + 28x$ where x is the number of units produced. Find 'x' so that the profit is maximum.

    [**HINTS,** $P(x)$ = Profit function = Revenue − Cost = $px - C(x) = 300x - 4x^2 - (500 + 28x)$ = etc.]

13. A firm produces c tones of output at a total cost C = ₹ $\left(\frac{1}{10}x^3 - 5x^2 + 10x + 5\right)$. At what level of output will the marginal cost and the average variable cost attain their respective minima?
14. Find the external points and the extreme values of the following functions
    (i) $y = x^2 - 6x + 5$ (ii) $y = 3x^3 - 5x^2 - 40x + 20$
    (iii) $y = 3x - x^2$ (iv) $y = x^3 + 5x^2 - 12 + 18$

ANSWERS

4. Yes, at $x = 0$, **6.** (i) 2, −1; (ii) 91, −34; (iii) −7, −8, **7.** Max at $x = 1$ and Max. value = $2\frac{1}{3}$; Min. at $x = 3$ and Min. value = 1 **9.** Max. at $x = 1$ and Max value = 0; Min. at $x = 3$ and min. value = −28; $x = 0$ is a point of inflexion. **10.** (i) (1,2) and (−1, −14). **11.** $\dfrac{150(2 - q^2)}{(q^2 + 2)^2} - 4$. **12.** (i) 75; (ii) x = 34 **13.** $\frac{50}{3}$ tonnes, 25 tonnes.

□□□

Unit – III

(Functions of two or more Independent Variables)

6. **Partial Differentiation**
(With Application in Economic Models)

Chapter 6

PARTIAL DIFFERENTIATION

1. INTRODUCTION

In simple differentiation, the process of differentiation is limited to the functions of one independent variable only, i.e. $y = f(x)$. But in Partial differentiation, the concept of differentiation of a function is extended to more than one independent variable i.e. $Z = f(x, y)$. Generally, at the time of analysis of the business and economic problems, we find that one factor is dependent on more than one factor. For example, in a production function, a firm's output is dependent on Land, Labour, Capital and Organisations. On the other hand, the price of a commodity is dependent on many variables viz. demand for the commodity, labour cost, competitive price maintenance cost, Insurance and Taxes etc. Hence, we have to bear in mind that in most of the business and economic applications a function depends on more than one independent factors, and that such a function can only be solved by applying the techniques of partial differentiation.

Here, we shall restrict our study to the case of two independent variables only. However, we can easily extend the case, where there are more than two variables are involved.

2. PARTIAL DERIVATIVES

Consider a function of two independent variables,

$$z = f(x, y)$$

Here, we might wish to study the rate at which z changes as x changes, if y is held constant. By keeping y as constant, $z = f(x, y)$ naturally becomes a function of x alone and so, the derivative of the function can be calculated with respect to x. This derivative is called **Partial derivative of z (or f) with respect to x** and is denoted by–

$$\frac{\partial z}{\partial x}, \frac{\partial}{\partial x}[f(x, y)], \qquad f_x(x, y) \text{ or } z_x(x, y)$$

Similarly, the partial derivative of z (or f) with respect to y is denoted by

$$\frac{\partial z}{\partial y}, \frac{\partial}{\partial y}[f(x, y)] \qquad f_y(x, y) \text{ or } z_y(x, y)$$

which gives the rate of change of z with respect to y keeping x as constant.

The sign '∂' is called as 'del' and is used to show that the expression to be differentiated contains more than one variable.

We can define partial derivatives with the help of limits as follows :

3. GEOMETRIC INTERPRETATION OF PARTIAL DERIVATIVE

A functional relationship for two independent variables is expressed in the form U = $f(x, y)$. Where x and y are the independent variables and U is a single dependent variable. As cited above the cost of producing a certain item may depend upon variables such as land, labour, capital and material. The functional relationship for two or more than two independent variables is denoted by U= $f(x, y, z, \ldots)$. The U = $f(x, y, z)$ is termed as function of three variables. For each value of x, y and z the dependent variable "U" assumes a single number as specified by the relation.

Suppose, $\quad U = f(x, y)$

$$= 1 + 3x^2 + 2y^3$$

If the independent variables are $x = 1$ and $y = 2$ the dependent variable U is,

$$U = f(1, 2) = 1 + 3(1)^2 + 2(2)^3 = 1 + 3 + 16 = 20$$

Let (x, y) be a function of two variables results a dependent variable U as U = $f(x, y)$. Here, the rate at which U changes is dependent on the variation of x and y, of particular importance are the rate of change of U with respect to x when y remains constant and the rate of change of U with respect to y when x remains constant.

Let, $\quad U = f(x, y)$

Then, the first order partial derivative of U with respect to x when y is held constant is defined by

$$\frac{\partial U}{\partial x} = \lim_{\Delta_x \to 0} \frac{f(x + \Delta_{x,y}) - f(x)}{\Delta_x} \quad \text{(If limit exists)}$$

Z = $f(x, y)$ is defined as a function of the independent variables and there exists one and only one value of Z in the range of f for each ordered pair of real numbers (x, y) in the domain of f.

By convention, Z is the dependent variable, x and y are the independent variables. The partial derivative of Z with respect to x measures the instantaneous rate of change of Z with respect to x while y is held constant. The partial derivative of Z with respect to y while x is held constant. It is expressed as $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x(x, y)$ or f_x or Z_x.

Partial derivative $\frac{\partial z}{\partial y}$ is written as $\frac{\partial f}{\partial y}$ or $f_y(x, y)$ or f_y or Z_y.

$$\frac{\partial z}{\partial x} = \lim_{\Delta_x \to 0} \frac{f(x + \Delta_x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta_y \to 0} \frac{f(x, y + \Delta_y) - f(x, y)}{\Delta y}$$

EXAMPLE : If $z = 2x^2 + 3xy$, then find

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(i) In order to find $\frac{\partial z}{\partial x}$, we have to held y as a constant and differentiate z with respect to x. Thus

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(2x^2 + 3xy - 5y^2) = 4x + 3y$$

Similarly, in order to find $\frac{\partial z}{\partial y}$, we have to hold x as a constant, and differentiate z with respect to y.

Thus

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[2x^2 + 3xy - 5y^2] = 3x - 10y$$

ILLUSTRATION 1. Find the first-order partial derivatives of the following functions :

(i) $2x^3y - xy^2 + y^4$ (ii) $\log(x^2 - y^2)$ (iii) $\frac{x^2}{x - y + 1}$ (iv) $y^2 e^{3x^2} + y^3$

SOLUTION

(i) To find $\frac{\partial z}{\partial x}$, we have to treat y as a constant and differentiate z with respect to x.

Thus, we have

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[2x^3y - xy^2 + y^4] = 2y\frac{\partial}{\partial x}(x^3) - y^2\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y^4) = 2y.3x^2 - y^2 + 0 = 6x^2y - y^2$$

Again, to find $\frac{\partial z}{\partial y}$, we have to treat x as a constant and differentiate z with respect to y.

Thus, we have

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[2x^3y - xy^2 + y^4] = 2x^3\frac{\partial}{\partial y}(y) - x.\frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(y^4) = 2x^3 - x.2y + 4y^3 = 2x^3 - 2xy + 4y^3$$

(ii) Taking y as a constant and differentiating z with respect to x, we get

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\log(x^2 - y^2)]$$

$$= \frac{1}{x^2 - y^2}\frac{\partial}{\partial x}(x^2 - y^2) = \frac{2x}{x^2 - y^2}$$

Again, by taking x as a constant and differentiating z with respect to y, we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\log(x^2 - y^2)]$$

$$= \frac{1}{x^2 - y^2}\frac{\partial}{\partial x}(x^2 - y^2) = \frac{-2y}{x^2 - y^2}$$

(iii) By considering y as a constant, the differentiation of z with respect to x will be,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2}{x - y + 1}\right] = \frac{(x - y + 1)(2x) - x^2(1)}{(x - y + 1)^2}$$

$$= \frac{2x(x - y + 1) - x^2}{(x - y + 1)^2}$$

Again, by considering x as a constant, the differentiation of z with respect to y will be,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2}{x - y + 1}\right]$$

$$= x^2 \frac{\partial}{\partial x}\left[\frac{1}{x-y+1}\right] = x^2\left[\frac{(x-y+1)\times 0 - 1(-1)}{(x-y+1)^2}\right]$$

$$= \frac{x^2}{(x-y+1)^2}$$

(iv) By considering y as a constant, the differentiation of z with respect to x will be

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(y^2 e^{5x^2+y^3})$$

$$= y^2 e^{5x^2+y^3} \cdot \frac{\partial}{\partial x}(5x^2+y^3)$$

$$= y^2 e^{5x^2+y^3} \cdot (10x) = 10xy^2 e^{5x^2+y^3}$$

Again, by considering x as a constant, the differentiation of z with respect to y will be

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(y^2 e^{5x^2+y^3})$$

$$= y^2 \cdot \frac{\partial}{dy} e^{5x^2+y^3} + e^{5x^2+y^3} \cdot \frac{\partial}{dy}(y)^2$$

$$= \left\{y^2 e^{5x^2+y^3} \cdot (3y^2)\right\} + \left\{e^{5x^2+y^3} \cdot (2y)\right\}$$

$$= 3y^4 \cdot e^{5x^2+y^3} + 2ye^{5x^2+y^3}$$

$$= (3y^3+2) \cdot ye^{5x^2+y^3}$$

ILLUSTRATION 2. (i) If $z = f(y/x)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

(ii) If $z = f(x-y)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

And verify that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

SOLUTION

(i) Treating y as a constant,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = y(-x^{-2}) = -\frac{y}{x^2}$$

Again, by treating x as a constant

We have, $$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}\times y^{1-1} = \frac{1}{x}$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\times\frac{-y}{x^2} + y\times\frac{1}{x} = -\frac{y}{x} + \frac{y}{x} = 0$$

Proved

(ii) Treating y as a constant,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$$

Again, by treating x as a constant,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$$

$$\therefore \quad \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1 + (-1) = 0$$

(Proved)

Remember :

The concept of Partial differentiation can be extended to the function of more than two variables in a natural way. For example if $z = f(x_1, x_2, x_3, \ldots\ldots, x_n)$ is a function of n variables then to find $\frac{\partial z}{\partial x_1}$, we have to consider the other variables viz. $x_2, x_3 \ldots\ldots x_n$ as constants and differentiate z with respect to x_1. To find $\frac{\partial z}{\partial x_2}$, we have to consider the variables $x_1, x_3, x_4 \ldots\ldots x_n$ as constants and differentiate z with respect to x_2 and so on.

ILLUSTRATION 3. If $u = \log(x^2+y^2+z^2)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2$$

SOLUTION

We have

$$\frac{\partial u}{\partial x} = \frac{\partial}{dx}[\log(x^2+y^2+z^2)] = \frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= \frac{1}{x^2+y^2+z^2}\times 2x = \frac{2x}{x^2+y^2+z^2}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2+z^2} \text{ and } \frac{\partial u}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\cdot\frac{2x}{x^2+y^2+z^2} + y\cdot\frac{2y}{x^2+y^2+z^2} + z\cdot\frac{2z}{x^2+y^2+z^2}$$

$$= \frac{2x^2}{x^2+y^2+z^2} + \frac{2y^2}{x^2+y^2+z^2} + \frac{2z^2}{x^2+y^2+z^2} = 2\left[\frac{x^2+y^2+z^2}{x^2+y^2+z^2}\right] = 2 \quad \text{(Proved)}$$

ILLUSTRATION 4. If $z = (x^2+y^2)/(x+y)$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

SOLUTION

We have, $z = \dfrac{x^2+y^2}{x+y}$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right) = \frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

And
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+y}\right) = \frac{(x+y).2y-(x^2+y^2).1}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$$

$$\text{L.H.S. } \left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}\right]^2$$

$$= \left[\frac{x^2-y^2+2xy-y^2+x^2-2xy}{(x+y)^2}\right]^2 = \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2 = 4\left[\frac{x-y}{x+y}\right]^2$$

$$\text{R.H.S. } 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = 4\left[1-\frac{x^2-y^2+2xy}{(x+y)^2}-\frac{y^2-x^2+2xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+y^2+2xy-x^2+y^2-2xy-y^2+x^2-2xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = 4\left[\frac{x-y}{x+y}\right]^2$$

$\therefore$ L.H.S. = R.H.S. (Proved)

Hence $\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)^2 = \left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)$

EXERCISE (A)

1. Find the first order partial derivatives of the following functions :

(i) $2x^2y+5xy-y^3$

(ii) $\log(x^2+y^2)$

(iii) $3x^2y^2+y^2$

(iv) $x^3+3x^2y-y^3$

(v) $\dfrac{x^2+y^2}{x^2+y}$

(vi) $\dfrac{y}{x}+\dfrac{x}{y}$

(vii) $y\,e^{-y/x}$

(viii) $e^{x^2}+y^2$

(ix) $(5x+4y)^3$

(x) $e^{x^2+xy+y^2}$

[Ans. (i) $4xy+5y;\ 2x^2-5x-3y^2$, (ii) $\dfrac{2x}{x^2+y^2};\dfrac{2y}{x^2+y^2}$, (iii) $6x^2y\ ;\ 6x^2y+2y$, (iv) $3x\ (x+2y)$; $3(x^2-y^2)$, (v) $\dfrac{2xy(1-y)}{(x^2+y)^2};\dfrac{x^2(2y-1)+y^2}{(x^2+y)^2}$, (vi) $-\dfrac{y}{x^2}+\dfrac{1}{y};\dfrac{1}{x}-\dfrac{x}{y^2}$, (vii) $\dfrac{y^2}{x^2}e^{-y/x};\ e^{-y/x}\left(1-\dfrac{y}{x}\right)$, (viii) $2x\,e^{x^2+y^2};\ 2y\,e^{x^2+y^2}$, (ix) $15(5x+4y)^2;\ 12(5x+4y)^2$, (x) $e^{x^2+xy+y^2}(2x+y);\ e^{x^2+xy+y^2}(x+2y)$]

2. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for each of the following functions :

(i) $z = e^{xy}+\log(xy)$

(ii) $z = xe^{x^2}-y^2$

(iii) $z = \dfrac{x^2+y^2}{xy}$

(iv) $z = x^2y\,e^y$

(v) $z = \log|\sqrt{x^2+y^2}|$

[Ans. (i) $ye^{xy}+\dfrac{1}{x};\ xe^{xy}+\dfrac{1}{y}$, (ii) $(2x^2+1)e^{x^2-y^2};-2xye^{x^2-y^2}$, (iii) $\dfrac{1}{y}-\dfrac{1}{x^2};\dfrac{x}{y^2}+\dfrac{1}{x}$, (iv) $2xy\ e^y;\ x^2(y+1)\,e^y$, (v) $\dfrac{x}{x^2+y^2},/\dfrac{y}{x^2+y^2}$]

3. From the following function, find f_x and f_y :

(i) $f(x, y) = 3x^2y^2+x^5+3y^2$

(ii) $f(x, y) = e^{x^y}$

(iii) $f(x, y) = \sqrt{xy}$

(iv) $f(x, y) = 5x\log(x^2+y)$

Ans. (i) $f_x = 6x^2+5x^4$, $f_y = 6x^2y+6y$, (ii) $f_x = y\ e^{y-1}\,e^{x^y}$, $f_y = x^y\log x\,e^{x^y}$, (iii) $f_x = \dfrac{1}{2}\sqrt{\dfrac{y}{x}}$, $f_y = \dfrac{1}{2}\sqrt{\dfrac{x}{y}}$ (iv) $f_x = \dfrac{10x^2}{x^2+y}+5\log(x^2+y)$, $f_y = 5x/x^2+y$

4. (i) If $z = 3x^4-x^2y^2+xy^3$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 4z$$

(ii) If $z = \log(y/x)$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 0$$

(iii) If $z = \log(e^x+e^y)$, prove that

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y} = 1$$

(iv) If $z = \dfrac{e^{xy}}{e^x+e^y}$, prove that

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y} = (x+y-1)^2$$

4. PARTIAL DERIVATIVES OF HIGHER ORDER

If $z = f(x, y)$, then partial derivatives of z with respect to x and y, will be $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively, and again, these are, in general, the functions of x and y.

Thus, we can process partial derivatives of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ just as we formed those of z to obtain the following second-order partial derivatives of z.

(i) $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$ (ii) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}$

(iii) $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}$ (iv) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$

In order to find out $\frac{\partial^2 z}{\partial x^2}\left(\text{and } \frac{\partial^2 z}{\partial y^2}\right)$, we have to take two successive derivatives, each time treating y and x as a constant respectively.

Similarly, to find $\frac{\partial^2 z}{\partial y \partial x}\left(\text{and } \frac{\partial^2 z}{\partial x\,\partial y}\right)$ we have to take two successive derivatives, whereas in the first differentiation x (respectively) y is treated as a constant and in the second differentiation of y (respectively) x is treated as a constant. The derivatives $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are called **mixed or cross partial derivatives.**

ILLUSTRATION 5. Find the second order partial derivatives of

$$z = 4x^2 + 9xy - 5y^2$$

SOLUTION

We have,
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x^2 + 9xy - 5y^2) = 8x + 9y,$$

Therefore,
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{dx}(8x+9y) = 8\times 1 + 9\times 0 = 8$$

And
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8x+9y) = 80+9\times 1 = 9$$

Similarly
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x^2 + 9xy - 5y^2) = 9x - 10y$$

Therefore,
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(9x-10y) = 0 - 10 = -10$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(9x-10y) = 9 - 0 = 9$$

Thus,
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 9, \frac{\partial^2 z}{\partial x^2} = 8 \text{ and } \frac{\partial^2 z}{\partial y^2} = -10.$$

ILLUSTRATION 6. If $z = \log(x^2+y^2)$, then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

SOLUTION

We have,
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^2+y^2)\right] = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{x^2+y^2}$$

$\therefore$
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2).2 - (2x).(2x)}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2}$$

Again,
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\log(x^2+y^2)\right] = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial y}(x^2+y^2) = \left(\frac{2y}{x^2+y^2}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2).2 - 2y.2y}{(x^2+y^2)^2} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

Now,
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} = 0 \text{ (Proved)}$$

ILLUSTRATION 7. If $u = \log(x^2+y^2+z^2)$, show that

$$x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$$

SOLUTION

We have,
$$u = \log(x^2+y^2+z^2)$$

$\therefore$
$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\log(x^2+y^2+z^2) = \frac{1}{x^2+y^2+z^2}\cdot\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \frac{2z}{x^2+y^2+z^2} \text{ (treating } x \text{ and } y \text{ as constant)}$$

And
$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{2z}{x^2+y^2+z^2}\right) = \frac{(x^2+y^2+z^2)0 - (2z)(2y)}{(x^2+y^2+z^2)^2}$$

$$= \frac{-4yz}{(x^2+y^2+z^2)^2}$$

Now $$x\frac{\partial^2 u}{\partial y\,\partial z} = x.\frac{-4yz}{(x^2+y^2+z^2)^2} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^2+y^2+z^2)\right] = \frac{2x}{x^2+y^2+z^2}$$

Again, $$\frac{\partial^2 u}{\partial_z \partial_x} = \frac{\partial}{\partial_z}\left(\frac{\partial u}{\partial_x}\right) = \frac{\partial}{\partial_z}\left(\frac{2x}{x^2+y^2+z^2}\right)$$

$$= \frac{(x^2+y^2+z^2)\times 0 - 2x.2z}{(x^2+y^2+z^2)^2} = \frac{-4xz}{(x^2+y^2+z^2)^2}$$

Now, $$y.\frac{\partial^2 u}{\partial_z \partial_x} = y\times\frac{-4xz}{(x^2+y^2+z^2)^2} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad ...(2)$$

And, $$\frac{\partial u}{\partial_y} = \frac{\partial}{\partial_y}[\log(x^2+y^2+z^2)] = \frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial_y}(x^2+y^2+z^2) = \frac{2y}{x^2+y^2+z^2}$$

∴ $$\frac{\partial^2 u}{\partial_x \partial_y} = \frac{\partial}{\partial_x}\left(\frac{2y}{x^2+y^2+z^2}\right) = \frac{(x^2+y^2+z^2)\times 0 - 2y.2x}{(x^2+y^2+z^2)^2} = \frac{-4xy}{(x^2+y^2+z^2)^2}$$

Now, $$z\frac{\partial^2 u}{\partial_x \partial_y} = z\times\frac{-4xy}{(x^2+y^2+z^2)^2} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad ...(3)$$

By comparing equations (1), (2) and (3), we have

$$x\frac{\partial u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y} = \frac{-4xyz}{(x^2+y^2+z^2)^2}$$

Hence $$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}$$

Remark : Each of the four second-order partial derivatives cited as above is a function of x and y, and by a further partial differentiation process, we can get eight partial derivatives of third order. We can thus, proceed to partial derivatives of fourth and any other higher orders.

5. HOMOGENEOUS FUNCTIONS

The polynomial function, where the degree of each term is n and

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n$$

is called *a* homogeneous function of degree n in x and y. Again,

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n$$
$$= x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + ... + + a_n(y/x)^n]$$
$$= x^nf(y/x)$$

Thus, *a* function $f(x, y)$ is said to be homogeneous function of degree n in x and y, if

$$\boxed{f(x,y) = x^n f(y/x)}$$

Definition : A function $z = f(x, y)$ is said to be homogeneous of degree n (n being a constant) if, for any real number λ ;

$$f(\lambda_x, \lambda_y) = \lambda^n f(x,y)$$

Thus, if both x and y are multipled by the same real number, then the resulting function value is a power of the number times the function value $f(x, y)$.

EXAMPLE: If $f(x,y) = 2x^2y + xy^2 - y^3$, then

$$f(\lambda_x, \lambda_y) = 2(\lambda x)^2(\lambda y) + (\lambda x)(\lambda y)^2 - (\lambda y)^3$$
$$= 2\lambda^3x^2y + \lambda^3xy^2 - \lambda^3y^3$$
$$= \lambda^3(2x^2y + xy^2 - y^3)$$
$$= \lambda^3 f(x, y)$$

6. EULER'S THEOREM

Let $z = f(x, y)$ be a homogeneous function in x and y of degree n, and having continuous partial derivatives, then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

Proof :

Since z is a homogeneous function of degree n, we can write

$$f(x, y) = x^n f(y/x)$$

Substituting (y/x) by t, we get, $f(x, y) = x^n f(t)$

Differentiating both the sides partially with respect to x and y respectively,

We have, $$\frac{\partial z}{\partial x} = nx^{n-1}f(t) + x^n f'(t)\left(\frac{\partial t}{\partial x}\right)$$

$$= nx^{n-1}f(y/x) + x^n f'(y/x)(-y/x^2) \quad \left(\because t = \frac{y}{x}\right)$$

$$= nx^{n-1}f(y/x) - x^{n-2}.y.f'(y/x)$$

Again, we have, $$\frac{\partial z}{\partial y} = f(t).\left(\frac{\partial}{\partial y}\right)(x^n) + x^n.f'(t).\frac{\partial t}{\partial y}$$

$$= f(t)\times 0 + x^n f'\left(\frac{y}{x}\right)\times\frac{1}{x} \quad \left(\because t = \frac{y}{x}\right)$$

$$= 0 + x^{n-1}f'\left(\frac{y}{x}\right) = x^{n-1}f'\left(\frac{y}{x}\right)$$

Multiplying $\frac{\partial z}{\partial x}$ by x and $\frac{\partial z}{\partial y}$ by y and then adding their products we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f(y/x) - x^{n-1}.y.f'(y/x) + x^{n-1}.y.f'(y/x)$$

$$= n.x^n f(y/x) = n f(x,y) = nz$$

Hence, $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{(Proved.)}$$

ILLUSTRATION 8. Verify Euler's theorem in the following cases :

(i) $z = x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4$ (ii) $z = \frac{x(x^3-y^3)}{x^3-y^3}$

SOLUTION

(i) We have $z = x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4$

The above function is a homogeneous function of x and y of degree 4. So by Euler's theorem, we must have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4z$

Verification:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4) = 4x^3 - 9x^2y + 10xy^2 + 4y^3$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4) = -3x^3 + 10x^2y + 12xy^2 - 8y^3$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x(4x^3 - 9x^2y + 10xy^2 + 4y^3) + y(-3x^3 + 10x^2y + 12xy^2 - 8y^3)$$

$$= 4x^4 - 9x^3y + 10x^2y^2 + 4xy^3 - 3x^3y + 10x^2y^2 + 12xy^3 - 8y^4$$
$$= 4x^4 - 12x^3y + 20x^2y^2 + 16xy^3 - 8y^4$$
$$= 4(x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4)$$

$= 4z$ (This verifies the Euler's theorem)

(ii) We have, $\frac{x(x^3-y^3)}{(x^3+y^3)}$ which is obviously a homogeneous function of x and y of degree 4–3 or i.e. 1.

Here, each term in the numerator is of degree 4 while each term in the denominator is of degree 3.

In order to verify Euler's theorem, we are to show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = z \quad (\because \text{ here, } n = 1)$$

Now, $\log z = \frac{x(x^3-y^3)}{(x^3+y^3)} = \log x + \log(x^3 - y^3) - \log(x^3 + y^3)$

or $\log z = \log x + \log(x^3 - y^3) - \log(x^3 + y^3)$

Differentiating partially both the side we have

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{1}{x} + \frac{3x^2}{x^3-y^3} - \frac{3x^2}{x^3+y^3} \quad ...(1)$$

$$\frac{1}{z}\frac{\partial z}{\partial y} = 0 - \frac{3y^2}{x^3-y^3} - \frac{3y^2}{x^3+y^3} = -\frac{3y^2}{x^3-y^3} - \frac{3y^2}{x^3+y^3} \quad ...(2)$$

Multiplying x in the equation (1) we get

$$\frac{1}{z}x.\frac{\partial z}{\partial x} = 1 + \frac{3x^3}{x^3-y^3} - \frac{3x^3}{x^3+y^3} \quad ...(3)$$

Multiplying y in the equation (2) we get

$$\frac{1}{z}y.\frac{\partial z}{\partial y} = \frac{-3y^3}{x^3-y^3} - \frac{3y^3}{x^3+y^3} \quad ...(4)$$

By adding the equations (3) and (4) we have,

$$\frac{1}{z}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = \left(1 + \frac{3x^3}{x^3-y^3} - \frac{3x^3}{x^3+y^3} - \frac{3y^3}{x^3-y^3} - \frac{3y^3}{x^3+y^3}\right)$$

$$= 1 + \frac{3(x^3-y^3)}{x^3-y^3} - \frac{3(x^3+y^3)}{x^3+y^3}$$

$$= 1 + 3 - 3 = 1$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

This verifies Euler's theorem

ILLUSTRATION 9. Verify Euler's theorem for the function $z = x^3 \log(y/x)$.

SOLUTION

We have, a homogeneous function $z = x^3 \log(y/x)$

Now, $\frac{\partial z}{\partial x} = \left(\log\frac{y}{x}\right)\frac{\partial}{\partial x}x^3 + x^3.\frac{\partial}{\partial x}\log\left(\frac{y}{x}\right)$

$$= \log\left(\frac{y}{x}\right).(3x^2) + x^3.\frac{x}{y}.\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= 3x^2.\log\left(\frac{y}{x}\right) + \frac{x^4}{y}\left(y \times \frac{-1}{x^2}\right) = 3x^2\log\left(\frac{y}{x}\right) - x^2$$

And $x\frac{\partial z}{\partial x} = x\left[3x^2.\log\left(\frac{y}{x}\right) - x^2\right] = 3x^3.\log\left(\frac{y}{x}\right) - x^3 \quad ...(1)$

Again $\frac{\partial z}{\partial x} = \log\left(\frac{y}{x}\right).\frac{\partial}{\partial y}(x^3) + x^3.\frac{\partial}{\partial y}\log\left(\frac{y}{x}\right)$

$$= \log\left(\frac{y}{x}\right) \times 0 + x^3\left(\frac{x}{y} \times \frac{1}{x}\right) = \frac{x^3}{y}$$

And $y\frac{\partial z}{\partial y} = x^3 \quad ...(2)$

By adding the equations (1) and (2) we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3x^3\log\left(\frac{y}{x}\right) - x^3 + x^3 = 3\left[x^3\log\left(\frac{y}{x}\right)\right] = 3z$$

ILLUSTRATION 10.

(a) If $v = \log\frac{x^3+y^3}{x^2+y^2}$, prove by Euler's theorem that $x\frac{\partial v}{\partial_x} + y\frac{\partial v}{\partial_y} = 1$

(b) If $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$, prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f(x, y, z)$$

SOLUTION

We have, $v = \log_e \frac{x^3+y^3}{x^2+y^2}$ i.e. $e^v = \frac{x^3+y^3}{x^2+y^2}$

If $z = e^v$, then $z = \frac{x^3+y^3}{x^2+y^2} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x^2\left(1+\frac{y^2}{x^2}\right)} = x.\left\{\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)^2}\right\}$

Thus, z is a Homogeneous function of x, y of degree 1.

Hence, by Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad (\because 1 \times z = z)$

Now, $z = e^v, \Rightarrow \frac{\partial z}{\partial x} = e^v.\frac{dv}{dx}$

and $\frac{\partial z}{\partial y} = e^v.\frac{dv}{dy}$

We have, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z, \Rightarrow x.e^v.\frac{\partial v}{\partial x} + y.e^v.\frac{\partial v}{\partial y} = e^v$

$\Rightarrow \quad e^v\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = e^v$

$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$ (Proved)

(b) We have, $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$

$$\frac{\partial f}{\partial x} = 3x^2 - 3yz. \frac{\partial f}{\partial y} = 3y^2 - 3xz \text{ and } \frac{\partial f}{\partial z} = 3z^2 - 3xy$$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(3x^2 - 3yz) + y(3y^2 - 3xz) + z(3z^2 - 3xy)$$

$$= 3x^3 - 3xyz + 3y^3 - 3xyz + 3z^3 - 3xy^2)$$
$$= 3(x^3 + y^3 + z^3 - 3xyz) = 3.[f(x, y, z)]$$

Thus, $x\frac{\partial f}{\partial_x} + y\frac{\partial f}{\partial_y} + z\frac{\partial f}{\partial_z} = 3f(x, y, z)$ (Proved)

ILLUSTRATION 11. If $z = f(x, y)$ is a homogeneous function of degree n, then show that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

SOLUTION

Since, z is a homogeneous function of degree n, by Euler's theorem

we have, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ ...(1)

Differentiating the equation (1) with respect to x keeping y as a constant, we get,

$$\left(x.\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\times 1\right) + \left(y.\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\times 0\right) = n\frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x}$$

Or $$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = (n-1)\frac{\partial z}{\partial x}$$

Multiplying both the sides of this equation by x we get,

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \partial y} = (n-1)x\frac{\partial z}{\partial x} \quad ...(2)$$

Now by differentiating the equation (1) with respect to y keeping x as a constant we get,

$$\left(x.\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\times 0\right) + \left(y.\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\times 1\right) = n\frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n\frac{\partial z}{\partial y}$$

Or $$x\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = (n-1)\frac{\partial z}{\partial y}$$

Multiplying both the sides of the above equation by y we get,

$$xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)y\frac{\partial z}{\partial y} \quad ...(3)$$

Adding the equations (2) and (3) we get,

$$\left(x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \partial y}\right) + \left(xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}\right)$$

$$= (n-1)x\frac{\partial z}{\partial x} + (n-1)y\frac{\partial z}{\partial y}$$

$$= (n-1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] = (n-1)\,n\,z = n(n-1)z$$

Hence, $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z \text{ (Proved)}$$

EXERCISE (B)

1. Find the four second order partial derivatives of the following functions :

(i) $f(x, y) = 2x^2y^3$ (ii) $f(x, y) = 2x^3 - 5xy + y^3$

(iii) $f(x, y) = xe^{xy} - y^2$ (iv) $f(x, y) = \log (x^2y^4)$

[**Ans.** (i) $f_{xx} = 4y^3, f_{xy} = 12xy^2, f_{yx} = 12xy^2, f_{yy} = 12x^2y$, (ii) $f_{xx} = 12x$, $f_{xy} = -5$, $f_{yx} = -5, f_{yy} = 6y$ (iii) $f_{xx} = (xy+2)xe^{xy}, f_{xy} = (xy+2)xe^{xy}$ $f_{yx} = (xy+2)xe^{xy}$, $f_{yy} = x^3e^{xy} - 2$, (iv) $f_{xx} = -\frac{2}{x^2}, f_{xy} = 0, f_{yx} = 0, f_{yy} = -\frac{4}{y^2}$]

2. (i) If, $z = x^3 + y^3 - 3x^2y$, then verify that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 y}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 6z$$

(ii) If, $z = \sqrt{ax^2 + 2hxy + by^2}$, then verify that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 y}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

3. **Verify Euler's theorem when**

(i) $z = x^2 + y^2$ (ii) $z = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(iii) $z = \frac{xy}{x+y}$ (iv) $z = x^n \log(y/x)$

(v) $z = x^3 + y^3 + x^2y$ (vi) $z = \frac{x^3 + y^3}{xy}$

4. (i) If, $z = \log\left(\frac{x^3 + y^3}{x - y}\right)x$, then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$

(ii) If, $z = \log\left(\frac{x^2 + y^2}{x + y}\right)$, then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$

7. TOTAL DIFFERENTIALS

Total differential is the sum of partial differentials arising from the separate variation of the variables. In other words, in a given function, Z = f(x, y) having continuous first partial derivative in a region of the XY-Plane.

Thus, the total differential is,

$$\partial z = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy$$

The partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the function $z = f(x, y)$ measure the small variations in x or y. The total differential provides us with an idea of linear approximation of the small variations in the function z due to small variations in both x and y.

The symbols dx and dy in the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ are called the differentials. The differential of y is defined as thus, $dy = f'(x)\,\Delta x$

Then, the differential of dx of x is given by $dx = 1\,\Delta x$ ...(1)

Since derivative of x with respect to itself is 1. Thus, the equation (1) can be written as

$$dy = f'(x)dx \quad ...(2)$$

This concept can be applied to the function of two, or more variables. Let $z = f(x, y)$ be a function of two variables x and y. For small increments Δx and Δy in x and y respectively, let Δz be the corresponding increment in z. By definition we have,

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right] \text{ and } \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

Further, $\lim_{\Delta x, \Delta y \to 0} \Delta z = dz$

Now $z + \Delta z = f(x + \Delta x, y + \Delta y)$

$$\Delta z = f(x + \Delta x, y + \Delta y) - z = f(x+\Delta x, y+\Delta y) - f(x, y)$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y+\Delta x) + f(x, y+\Delta y) - f(x, y)$$

$$\Delta z = \left[\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right]\Delta x + \left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]\Delta y$$

Taking limits on both the sides as $\Delta x \to$ and $\Delta y \to$ we have,

$$\lim_{\substack{\Delta x \to 0\\ \Delta y \to 0}} \Delta_z = \lim_{\substack{\Delta x \to 0\\ \Delta y \to 0}}\left\{\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right\}\lim_{\Delta x \to 0} \Delta x + \lim_{\substack{\Delta y \to 0\\ \Delta x \to 0}}\left\{\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right\}\lim_{\Delta y \to 0} \Delta y$$

Let dx, dy and dz be the limiting values of Δx, Δy and Δz. Then, from the above equation we get,

$$\boxed{dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy}$$

In the above equation the terms $\frac{\partial z}{\partial x} \times dx$, $\frac{\partial z}{\partial y} \times dy$ are called **partial differentials of z with respect to x and y respectively. The sum of these partial differentials of a function is called the total differential.**

The result of the above equation can be extended to the functions of several independent variables. Thus, when $z = f(x, y, r)$ then,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y} \times dy + \frac{\partial z}{\partial r}dr$$

ILLUSTRATION 12. Find the total differentials of the function $z = f(x, y) = 2x^2y + y^3$

SOLUTION

We have, $\dfrac{\partial z}{\partial x} = 4xy$ and $\dfrac{\partial z}{\partial y} = 2x^2 + 3y^2$

Hence, the total differential of z is

$$dz = \frac{\partial z}{\partial x} \times dx + \frac{\partial z}{\partial y} \times dy = (4xy)\, dx + (2x^2 + 3y^2)\, dy$$

ILLUSTRATION 13. If $z = \log\left\{\dfrac{(x^2 + y^2)}{xy}\right\}$, find dz.

SOLUTION

We have, $z = \log(x^2 + y^2) - \log x - \log y$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x} = \frac{2x^2 - x^2 - y^2}{x(x^2 + y^2)} = \frac{x^2 - y^2}{x(x^2 + y^2)}$$

And

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y} = \frac{2y^2 - x^2 - y^2}{y(x^2 + y^2)} = \frac{y^2 - x^2}{y(x^2 + y^2)}$$

Now

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \frac{x^2 - y^2}{x(x^2 + y^2)} dx + \frac{y^2 - x^2}{y(x^2 + y^2)} dy = \frac{x^2 + y^2}{x(x^2 + y^2)} \cdot \frac{y}{y} dx + \frac{y^2 - x^2}{y(x^2 + y^2)} \cdot \frac{x}{x} dy$$

$$= \frac{x^2 - y^2}{x(x^2 + y^2)} \cdot \frac{y}{y} dx + \frac{y^2 - x^2}{y(x^2 + y^2)} \cdot \frac{x}{x} dy$$

$$= \frac{x^2 - y^2}{xy(x^2 + y^2)} (y\, dx - x\, dy) \quad \textbf{Ans.}$$

ILLUSTRATION 14. Find the total $\dfrac{dz}{dt}$ derivative of the following:

$$z = 3x^2 - 2xy + 5y \text{ if } x = 3t^2 + 2t,\ y = 5t + 7$$

SOLUTION

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = 6x - 2y \text{ and } \frac{\partial z}{\partial y} = -2x + 5$$

$$x = 3t^2 + 2t,\ \frac{dx}{dt} = 6t + 2, \text{ and } y = 5t + 7,\ \frac{dy}{dt} = 5$$

$$\frac{dz}{dt} = (6x - 2y)(6t + 2) + (-2x + 5) \times 5$$

$$= 36\,xt - 12\,yt + 2x - 4y + 25 \quad \textbf{Ans.}$$

ILLUSTRATION 15. Find the total derivative of z w.r.t. x where

$$z = 3x^3 + 2xy + 4y^2 \text{ and } y = 3x + 5$$

SOLUTION

$$dz = f_x\, dx + f_y dy$$

$$\frac{dz}{dx} = f_x \frac{dx}{dx} + f_y \frac{dy}{dx} \text{ or } \frac{dz}{dx} = f_x + f_y \frac{dy}{dx}$$

$$\frac{\partial z}{\partial x} = f_x = 9x^2 + 2y,\ \frac{\partial z}{\partial y} = f_y = 2x + 8y \text{ and } \frac{dy}{dx} = 3$$

$$\frac{dz}{dx} = (9x^2 + 2y) + (2x + 8y) \times 3$$

$$= 9x^2 + 26y + 6x. \quad \textbf{Ans.}$$

ILLUSTRATION. 16. Find the total derivative $\dfrac{dz}{dt}$ of the function given below.

$$z = x^2 + 2x + y^2, \text{ if } y = \frac{1}{x}, x = t$$

SOLUTION

$$f_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2x + y^2) = 2x + 2 + 0 = x^2 + 2x$$

$$f_y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2x + y^2) = 0 + 0 + 2y = 2y$$

$$\frac{dx}{dt} = \frac{d}{dt}(t) = 1 \text{ and } \frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$$

Now,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= (x^2 + 2x) \times 1 + \left(2y \times -\frac{1}{t^2}\right) = x^2 + 2x - \frac{2y}{t^2} \quad \textbf{Ans.}$$

8. DERIVATIVE OF AN IMPLICIT FUNCTION

Let two variables x and y are related to a given implicit relation $f(x, y) = 0$. In general, this defines y as a multi-valued function of x. However, the simplest way of dealing with such a function is to introduce a third variable z as under :

$$z = f(x, y)$$

However, $z = 0$ for all x and y related by $f(x, y) = 0$. Thus the given implicit function can be studied by relating those values of x and y which makes z equal to zero in a explicit function

$$z = f(x, y).$$

As x and y vary in any way, independently or not, the complete differential of z is given by

$$dz = f_x\,dx + f_y\,dy$$

If, (x, y) are values showing $z = 0$ and if, dx and dy are variations from these values keeping $z = 0$, then $z = 0$.

Thus, $f_x\,dx + f_y\,dy = 0$

Hence,

$$\boxed{\frac{dy}{dx} = -\frac{fx}{fy}}$$

ILLUSTRATION 17. If, $x^y + y^x = a^b$, find $\dfrac{dy}{dx}$.

SOLUTION

Given, $x^y + y^x = a^b$

or $x^y + y^x$

Writing $f(x, y) = x^y + y^x - a^b$ we have

$$f_x = y \cdot x^{y-1} + y^x \log y$$

And $f_x = x^y \log x + xy^{x-1}$

$$\therefore \quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

EXERCISE (C)

1. Find the total differentials of the following functions.

(i) $z = x^3 + y^3 + 3xy$ (ii) $z = e^{x^2+y^2}$

(iii) $z = e^{y/x}$ (iv) $z = \log\left(\dfrac{x+y}{x-y}\right)$

[**Ans.** (i) $(3x^2 + 3y)dx + (3y + 3x)\,dy$, (ii) $2e^{x^2+y^2}(x\,dx + y\,dy)$,

(iii) $\dfrac{e^{y/x}}{x^2}(-y\,dx + x\,dy)$, (iv) $\dfrac{2}{x^2 - y^2}(-y\,dx + x\,dy)$]

2. Find $\dfrac{dy}{dx}$ from the following implicit functions :

(i) $x^2y - x + 2y = 0$ (ii) $x^5 + y^5 = 5ax^2\,y^2$

[**Ans.** (i) $\dfrac{1-2xy}{x^2+2}$, (ii) $\dfrac{-x(x^3 - 2ay^2)}{y(y^3 - 2ax^2)}$]

APPLICATION IN ECONOMICS

ELASTICITY FUNCTION :

The technique of differential calculus is very much useful for analysis and interpretations of various economic problems. Some of the economics models relating to elasticity function and its application are explained here as under :

1. ELASTICITY

The term elasticity in economics means changeability, or adjustability of a certain factor in response to a change in some other related factor. For example, changeability of supply or demand in response to a change in the price is called elasticity of supply, and elasticity of demand respectively.

The elasticity of a function, $y = f(x)$ at the point x is the rate of proportional change in y per unit of proportional change in x. This is given by $\dfrac{Ey}{Ex} = \dfrac{x}{y}\dfrac{dy}{dx}$.

Price Elasticity of Supply

By price elasticity of supply we mean a relative change in the supply in response to a relative change in the price. In other words, it is the ratio of percentage change in quantity supplied to a percentage change in the price.

Thus, $$\text{P.E.S.} = \frac{\text{Percentage change in the quantity supplied}}{\text{Percentage change in the price}}$$

Symbolically, it is obtained by $\boxed{\eta_S = \dfrac{p}{x}\cdot\dfrac{dx}{dp}}$

Where, η_S denotes price elasticity of supply, p the price, and x the supply.

Note. Since price and supply go in the same direction η_S remains always +ve.

The symbol η is read as eta :

Price Elasticity of Demand

By price elasticity of demand we mean a proportionate change in the quantity demanded in response to a proportionate change in the price. In other words, it is the ratio of percentage change in the quantity demanded to a percentage change in the price.

Thus, $$\text{P.E.D.} = \frac{\text{Percentage change in the quantity demanded}}{\text{Percentage change in the price}}$$

Or $$= \frac{\text{Marginal quantity demanded}}{\text{Average quantity demanded}}$$

Symbolically, it is obtained by $\boxed{\eta_d = \dfrac{-p}{x}\cdot\dfrac{dx}{dp}}$

Where, η_d denotes price elasticity of demand, $\dfrac{x}{p}$ the average function of demand. $\dfrac{dx}{dp}$ the marginal function of demand and $|\eta_d|$ the absolute value of η_d.

Since, price and demand go in opposite direction, the value of ηd remains always –ve.

Point Elasticity of Demand

This is given by $$|\eta_d| = \frac{p}{x}\cdot\frac{dx}{dp}$$

The other notation of point elasticity of demand are Ed, and e.

The following illustrations will show how various elasticities are determined.

ILLUSTRATION 1. Find (i) Elasticity, and (ii) Elasticity when $x = 8$ for the function $y = 3x - 6$.

SOLUTION

Given, $y = 3x - 6$

(i) Elasticity

The elasticity of the above function would be obtained by

$$\frac{Ey}{Ex} = \frac{x}{y}\frac{dy}{dx} = \frac{x}{(3x-6)}\cdot\frac{d}{dx}(3x-6) = \frac{x}{3(x-2)}.3 = \frac{x}{x-2}$$

(ii) Elasticity when x = 8

The elasticity of the given function when $x = 8$ would be obtained by

$$\frac{Ey}{Ex} = \frac{x}{y}\cdot\frac{dy}{dx} = \frac{x}{(3x-6)}\cdot\frac{d}{dx} = \frac{x}{3(x-2)}.3 = \frac{8}{8-2} = \frac{4}{3}$$

The above result indicates that if x is increased by 1% then y increases by 4/3%.

ILLUSTRATION 2. The supply of certain goods is given by $x = a\sqrt{p-b}$, where x is the quantity supplied, p (*i.e.* $> b$) is the price, and a and b are +ve constants. Determine the expression for the elasticity of supply as a function of price, and using the calculus show that the elasticity decreases as the price increases, and becomes unity as the price equal to $2b$.

SOLUTION

We have, $$\eta_s = \frac{p}{x}\cdot\frac{dx}{dp}$$

Where, $$\frac{dx}{dp} = a.\frac{1}{2}(p-b)^{-1/2}.1 = \frac{a}{2\sqrt{p-b}}$$

And $$\frac{p}{x} = \frac{p}{a\sqrt{p-b}}$$

Thus, $$\eta_s = \frac{p}{a\sqrt{p-b}}\cdot\frac{a}{2\sqrt{p-b}} = \frac{p}{2\sqrt{p-b}}$$

Differentiating η_s w.r.t. p we get,

$$\frac{d}{dp}(\eta_s) = \frac{1}{2}\left[\frac{(p-b).1-p.1}{(p-b)^2}\right] = \frac{-b}{2(p-b)^2}, \quad \text{i.e. –ve} \quad [\because b > a]$$

Here, η_s is a decreasing function of p.

In other words, elasticity decreases as price increases

Again, when $$p = 2b, \eta_s = \frac{2b}{2(2p-b)} = \frac{b}{b} = 1 \text{ Proved.}$$

ILLUSTRATION 3. If η_1 and η_2 be the price elasticities of the supply function, $p = e^x$, $p = \frac{e^x}{x}$ respectively, then show that $\eta_1 . \eta_2 = \eta_2 - \eta_1$.

SOLUTION

For η_1 we are to differentiate both the sides of the equation $p = ex$.

Thus, $$\frac{dp}{dx} = \frac{d}{dx}(e^x) = e^x \qquad \therefore \frac{dx}{dp} = \frac{1}{e^x}$$

We have, $$\eta_1 = \frac{p}{x}\cdot\frac{dx}{dp} = \frac{p}{x}\cdot\frac{1}{dp/dx} = \frac{p}{x}\cdot\frac{1}{e^x} = \frac{e^x}{x}\cdot\frac{1}{e^x} = \frac{1}{x}$$

Similarly, for η_2 we are to differentiate both the sides of the equation $p = \frac{e^x}{x}$.

Thus, $$\frac{dp}{dx} = \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{xe^x - e^x.1}{x^2} = \frac{e^x(x-1)}{x^2}$$

$$\Rightarrow \quad \frac{dx}{dp} = \frac{x^2}{e^x(x-1)}$$

We have, $$\eta_2 = \frac{p}{x}\cdot\frac{dx}{dp} = \frac{e^x}{x.x}\cdot\frac{x^2}{e^x(x-1)} = \frac{1}{x-1}$$

Thus, $$\eta_1\eta_2 = 1\frac{1}{x}\left(\frac{1}{x-1}\right) = \frac{1}{x(x-1)} \qquad \text{...(1)}$$

And $$\eta_2 - \eta_1 = 1\frac{1}{x-1} - \frac{1}{x} = \frac{x-(x-1)}{x(x-1)} = \frac{1}{x(x-1)} \qquad \text{...(2)}$$

Now, from the above two equations (1) and (2) we find that $\eta_1 . \eta_2 = \eta_2 - \eta_1$

ILLUSTRATION 4. Determine the price elasticity of demand for the function,

$$x = 32 - 4p - p^2, \quad \text{where } p = 3.$$

SOLUTION

Given, $x = 32 - 4p - p^2$

Thus when, $p = 3, x = 32 - 4(3) - (3)^2 = 32 - 12 - 9 = 11$

We have, $$\frac{dx}{dp} = \frac{d}{dp}(32 - 4p - p^2) = -4 - 2p$$

When $$p = 3, \frac{dx}{dp} = -4 -2\,(3) = -4 - 6 = -10$$

The price elasticity of demand is given by $\eta_d = \frac{-p}{x}\cdot\frac{dx}{dp}$

$$= \frac{-3}{11}(-10) = \frac{30}{11} = 2.727 \text{ approx.}$$

Hence, the required Price elasticity of demand is 2.727.

ILLUSTRATION 5. Find the point elasticity of demand for the demand function

$$p = \frac{k}{x} \text{ (Where } k > 0).$$

SOLUTION

Given $\quad p = \frac{k}{x}$

Thus, $\quad \frac{dp}{dx} = \frac{d}{dx}(k/x) = k\frac{d}{dx}(x^{-1}) = -kx^{-2} = -\frac{k}{x^2}$

$\therefore \quad \frac{dx}{dp} = \frac{-x^2}{k}$

The point elasticity of demand is given by

$$|\eta_d| = \frac{-p}{x}\frac{dx}{dp} = \frac{-x}{x^2}\cdot\frac{-x^2}{k} = 1$$

Hence, the required point elasticity of demand = 1 for all $x > 0$.

ILLUSTRATION 6. Determine the elasticity of demand for the demand function, $x = \frac{27}{p^3}$, where x is the demand for goods at the price p.

SOLUTION

Given $\quad x = \frac{27}{p^3}$

Thus, the marginal quantity demanded, or $\frac{dx}{dp} = \frac{d}{dp}(27/p^3)$

$$= 27\frac{d}{dp}(p^{-3}) = -81p^{-4} = \frac{-81}{p^4}$$

Again, the average quantity demanded, or $\frac{x}{p} = \frac{27}{p^3} \times \frac{1}{p} = \frac{27}{p^4}$

The elasticity of demand is obtained by

$$|\eta_d| = \frac{-p}{x}\frac{dx}{dp} = \frac{-p^4}{27}\cdot\frac{-81}{p^4} = 3$$

Hence, the required elasticity of demand = 3

ILLUSTRATION 7. Find the elasticity of demand in terms of x for the demand function,

$$p = \frac{10}{(x+1)^2}.$$

SOLUTION

The elasticity of demand is obtained by $\eta_d = \frac{-p}{x}\cdot\frac{dx}{dp}$

Given $\quad p = \frac{10}{(x+1)^2}$ and $\frac{p}{x} = \frac{10}{(x+1)^2 . x}$

Again $\quad \frac{dp}{dx} = \frac{d}{dx}[10(x+1)^{-2}] = 10\frac{d}{dx}(x+1)^{-2}$

$$= 10[-2(x+1)^{-3}.\frac{d}{dx}(x+1)] = -20(x+1)^{-3}.1 = \frac{-20}{(x+1)^3}$$

$\therefore \quad \frac{dx}{dp} = \frac{-(x+1)^3}{20}$

Thus, $\quad \eta_d = \frac{-p}{x}\cdot\frac{dx}{dp} = \frac{-10}{(x+1)^2 . x}\cdot\frac{-(x+1)^3}{20} = \frac{(x+1)}{2x}$

Hence, the required elasticity of demand = $\frac{(x+1)}{2x}$.

ILLUSTRATION 8. Determine η_d for the demand function, $x = 50 + p + p^2$, where x is the demand for the commodity at the price p. Also, find the point elasticity at $p = 5$.

SOLUTION

(i) The elasticity of demand is given by $\eta_d = \frac{-p}{x}\cdot\frac{dx}{dp}$

Given $x = 50 + p + p^2 \therefore \frac{x}{p} = \frac{50 + p + p^2}{p} \quad \therefore \frac{p}{x} = \frac{p}{50 + p + p^2}$

Again, $\quad x = 50 + p + p^2$, thus, $\frac{dx}{dp} = \frac{d}{dp}(50 + p + p^2) = 1 + 2p$

Thus, $\quad \eta_d = \frac{-p}{x}\cdot\frac{dx}{dp} = \frac{-p}{50 + p + p^2}.(1 + 2p) = \frac{-p(1 + 2p)}{50 + p + p^2}$

(ii) Point elasticity at $p = 5$

We have, $\quad \eta_d = \frac{-p(1 + 2p)}{50 + p + p^2} \Rightarrow |\eta_d| = \frac{p(1 + 2p)}{50 + p + p^2}$

Putting 5 for p in the above we have,

$$|\eta_d| = \frac{5[1 + 2(5)]}{50 + 5 + (5)^2} = \frac{5 \times 11}{80} = \frac{55}{80}$$

$\therefore$ The point elasticity at $p = 5$ is $\frac{55}{80}$ or $\frac{11}{16}$, i.e. < 1

The above result shows that the demand at $p = 5$ is inelastic.

ILLUSTRATION 9. If the demand function for a product be $p = 4 - 5x^2$, then for what value of x does the demand have a unit elasticity ?

SOLUTION

Given $\quad p = 4 - 5x^2, \Rightarrow \dfrac{p}{x} = \dfrac{4-5x^2}{x}$

$\therefore \quad \dfrac{dp}{dx} = \dfrac{d}{dx}(4-5x^2) = -5.2x = -10x, \Rightarrow \dfrac{dx}{dp} = \dfrac{-1}{10x}$

We have $\quad \eta_d = \dfrac{-p}{x}.\dfrac{dx}{dp} = \dfrac{-(4-5x^2)}{x}.\dfrac{-1}{10x} = \dfrac{4-5x^2}{10x^2}$

A demand has a unit elasticity when $\eta_d = 1$

$\Rightarrow \quad \dfrac{4-5x^2}{10x^2} = 1, \Rightarrow 4 - 5x^2 = 10x^2, \Rightarrow 15x^2 = 4 \therefore x = \dfrac{\pm 2}{\pm\sqrt{15}} = \dfrac{2}{\sqrt{15}}$

Hence, the required value of $x = \dfrac{2}{\sqrt{15}}$ (ignoring the –ve sign as the quantity x can not be –ve)

ILLUSTRATION 10. A demand function is given by $xp^{\eta} = k$, where η and k are the constants. Determine the price elasticity of demand.

SOLUTION

Given $\quad xp^{\eta} = k, \Rightarrow x = kp^{-\eta}$

$\therefore \quad \dfrac{dx}{dp} = \dfrac{d}{dp}(kp^{-\eta}) = -\eta\, kp^{-\eta-1}$

We have, $\quad |\eta_d| = \dfrac{-p}{x}.\dfrac{dx}{dp} = \dfrac{-p}{kp^{-\eta}}.-\eta kp^{-\eta-1} = \dfrac{\eta kp^{-\eta}}{kp^{-\eta}} = \eta$

Hence, the required elasticity of demand is η at all the levels of price.

ILLUSTRATION 11. Prove that the elasticity of demand at all the points on the curve $xy = x^2$ will be numerically equal to one.

SOLUTION

Given $\quad xy = x^2, \Rightarrow x = \dfrac{x^2}{y} = x^2y^{-1}$

$\therefore \quad \dfrac{dx}{dy} = \dfrac{d}{dy}(x^2y^{-1}) = -x^2y^{-2}$

We have, $\quad |\eta_d| = \dfrac{-y}{x}.\dfrac{dx}{dy} = \dfrac{-y}{x^2y^{-1}} \times (-x^2y^{-2}) = 1$

Hence, the elasticity of demand at all the points on the curve is unity.

ILLUSTRATION 12. The demand for a commodity is given by $x = 48 - 3p^2$. Find the point elasticity of demand when $p = 3$. If the price of 3 is decreased by 5%, determine the relative increase in demand, and hence an approximation to the elasticity of demand.

SOLUTION

Given $\quad x = 48 - 3p^2$

$\Rightarrow \quad \dfrac{dx}{dp} = \dfrac{d}{dp}(48 - 3p^2) = -6p$

We have, $\quad \eta_d = \dfrac{-p}{x}.\dfrac{dx}{dp} = -\left(\dfrac{p}{48-3p^2}\right).(-6p) = \dfrac{6p^2}{48-3p^2}$

When $\quad p = 3, \eta_d = \dfrac{6(3)^2}{48-3(3)^2} = \dfrac{54}{21} = \dfrac{18}{7} = 2.57$

Thus, the point elasticity of demand when $p = 3$ is 2.57

Again, given old price = 3, $\Rightarrow$ old demand = 48 – 3(3)2 = 21

Now, decrease in the price = 5% = $\dfrac{5}{100} \times 3 = 0.15$

$\therefore$ The new price = 3 – 0.15 = 2.85

$\Rightarrow$ The new demand = 48 – 3 (2.85)2 = 48 – 3(8.1225) = 23.6325

Thus, the percentage increase in the demand

$$= \frac{\text{New demand} - \text{Old demand}}{\text{Old demand}} \times 100$$

$$= \frac{23.6325 - 21}{21} \times 100 = \frac{2.6325}{21} \times 100 = 12.54$$

We have, $\quad \eta_d = \dfrac{\text{Percentage Change in the quantity demanded}}{\text{Percentage change in the price}} = \dfrac{12.54}{5} = 2.51$

Hence, the required approximation to the elasticity of demand = 2.5

Income Elasticity of Demand

A change in the quantity demanded in response to a change in the income is called income elasticity of demand.

Algebraically, this is obtained by $\boxed{\eta_y = \dfrac{y}{x}.\dfrac{dx}{dy}}$, where x is the quantity demanded, and y is the income per head in the relevant group of the people.

Thus, $\quad$ if $\eta_y > 1$, the goods are of luxury type.

$\quad$ if $0 < \eta_y < 1$, the goods are of necessity type.

And $\quad$ if $\eta_y < 0$, goods are of inferior type.

Logarithmic Elasticity of Demand

This is obtained by $\boxed{\eta_d = \dfrac{-d(\log x)}{d(\log p)}}$ $\quad \left[\because \eta_d = \dfrac{-p}{x}.\dfrac{dx}{dp} = \dfrac{-\frac{1}{x}\frac{dx}{dp}}{1/p} = \dfrac{\frac{-d}{dp}(\log x)}{\frac{d}{dp}(\log p)} = \dfrac{d(\log x)}{d(\log p)}\right]$

2. EQUILIBRIUM OF A FIRM AND CONSUMERS

The equilibrium of a firm and consumers, or the market equilibrium as it is called, is a point of value (i.e. price × quantity) where a firm's supply (of goods) equals the consumer's demand (for the goods).

Thus, the point where a demand and a supply curve intersect is called **the equilibrium point** vide the graph displayed as under.

In the above graph, the point of intersection $(x_0\ ,\ p_0)$ is the equilibrium point, the quantity x_0 is the equilibrium quantity, and the price p_0 is the equilibrium price.

The equilibrium point, price, and quantity can also be algebraically resolved by equating the supply and demand functions or by computing the excess of the demand over the supply as illustrated below :

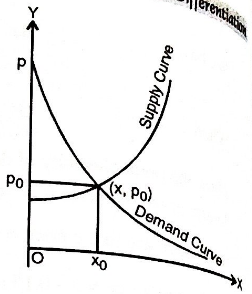


ILLUSTRATION 13. Determine the equilibrium point, price and quantity using both the methods from the following supply and demand functions :

$$Q_s = 0.2 + 0.07\,p, \qquad Q_d = 2 - 0.02p.$$

SOLUTION

***(i)* Under the Method of Equation**

At the equilibrium point we have, $Q_s = Q_d$

$\Rightarrow \quad 0.2 + 0.07p = 2 - 0.02p, \Rightarrow 0.09p = 1.8, \therefore p = \dfrac{1.8}{0.09} = 20$

Putting p at 20 in the supply function we get, $q = 0.2 + 0.07(20)$

$= 0.2 + 1.4 = 1.6$

Hence, equilibrium price = 20, quantity = 1.6, and point = (1.6, 20)

***(ii)* Under the method of Excess Demand**

Excess of demand over supply $= Q_d - Q_s$

$= (2 - 0.02p) - (0.2 + 0.07p) = 2 - 0.02p - 0.2 - 0.07p$

$= 1.8 - 0.09p, \Rightarrow p = \dfrac{1.8}{0.09} = 20$

Substituting the above value of p in the demand function we get,

$q = 2 - 0.02(20) = 2 - 0.4 = 1.6$

Hence, the equilibrium price = 20, quantity = 1.6, and point = (1.6, 20)

ILLUSTRATION 14. Find the equilibrium point, if the supply and demand functions of a product respectively are; $Q_S = 2p - 8$, $Q_d = 300 - 2p$

SOLUTION

At the equilibrium point we have, $Q_s = Q_d$

$\therefore \quad 2p - 8 = 300 - 2p, \Rightarrow 4p = 308, \therefore p = 77$

Substituting the above value of P in the demand or supply function we get,

$Q_d = 300 - 2(77) = 300 - 154 = 146$

Or $\quad Q_S = 2(77) - 8 = 154 - 8 = 146$

Hence, the equilibrium price or $p = 77$ and quantity or $x = 146$.

Thus, the required equilibrium point = (146, 77).

ILLUSTRATION 15. Using the method of excess demand, determine the equilibrium point from the following functions :

$$Q_s = 10 + \frac{2p}{3}, \text{ and } Q_d = 50 - \frac{8p}{7}$$

SOLUTION

By the method of excess demand we have, Quantity Demanded – Quantity Supplied.

Thus, $\quad$ Excess demand $= Q_d - Q_s = \left(50 - \dfrac{80}{7}\right) - \left(10 + \dfrac{20}{3}\right)$

$= 50 - \dfrac{8p}{7} - 10 - \dfrac{2p}{3} = 40 - 1.8\,p$

At the equilibrium point $Q_d = Q_s$ or $Q_d - Q_s = 0$, thus $40 – 1.8\,p = 0$

$\Rightarrow \quad p = \dfrac{40}{1.8} = 22.2$

Putting p at 22.2 in the supply function, $q = 10 + \dfrac{20}{3}$ we get,

$q = 10 + \dfrac{2(22.2)}{3} = 10 + \dfrac{44.4}{3} = 10 + 14.8 = 24.8$

Thus, the equilibrium price = 22.2 and quantity = 24.8

Hence, the equilibrium point = (24.8, 22.2)

ILLUSTRATION 16. The supply and demand function of a firm are respectively as under:

$$p = 18.2x - x^2, \text{ and } p = 2x - 3$$

Where p is the price, and x is the quantity.

Determine the equilibrium point.

SOLUTION

At the equilibrium point we have, $Q_s = Q_d$

$\Rightarrow \quad 18 - 2x - x^2 = 2x - 3, \Rightarrow x^2 + 4x - 21 = 0$

$\Rightarrow \quad x^2 + 7x - 3x - 21 = 0, \quad \Rightarrow x(x + 7) - 3(x + 7) = 0$

$\Rightarrow \quad (x - 3)(x + 7) = 0, \Rightarrow x = 3$, or $x = -7$

x being a quantity can not be –ve. $\therefore x = 3$.

Putting x at 3 in the demand function $p = 2x - 3$ we get, $p = 2(3) - 3 = 3$.

Hence, the required equilibrium point = (3, 3).

ILLUSTRATION 17. Given the following function for the two related markets, A and B, find the equilibrium conditions for each market, and the price for each of them :

$Q_d(A) = 82 - 3\,p_{(A)} + p(B), \qquad Q_S(A) = -5 + 15\,p_{(A)}$

$Q_d(B) = 92 + 2\,p_{(A)} - 4p(B), \qquad Q_S(B) = -6 + 32\,p_{(B)}$

SOLUTION

The equilibrium conditions are :

$Q_d(A) = Q_S(A), \Rightarrow 82 - 3p_{(A)} + p_{(B)} = -5 + 15p_{(A)} \Rightarrow 18\,p_{(A)} - p_{(B)} = 87$

And $\quad Q_d(B) = Q_S(B), \Rightarrow 92 + 2p_{(A)} - 4p_{(B)} = -6 + 32\,p_{(B)}, \Rightarrow 2\,p_{(A)} - 36\,p_{(B)} = 98$

or $\quad 18\,p_{(A)} - p_{(B)} = 87 \quad$ ...(1)

$2\,p_{(A)} - 36\,p_{(B)} = 98 \quad$ ...(2)

Solving the above two equations simultaneously we get,

$p_{(A)} = 5$, and $p_{(B)} = 3$

Thus, the equilibrium price in the market A = 5, and in the market B = 3

MARKET EQULIBRIUM (MR – MC APPROACH)

Market is known as a place where goods and services are purchased and sold. But in economics, Market is a contract between buyer and seller for buying and selling of the goods at the given price.

A firm is said to be in equilibrium when it maximized its profit. It is also called as the difference between total Revenue (R) and total Cost (C). The firm gives various outputs; sometimes it gives low and sometimes it gives high output which provides lower point to firm. When the situation is of nor high nor low i.e. Equilibrium is obtained and it gives more profit.

Once, the firm attained equilibrium the firm doesn't have the incentive to change its price and output because profit is already maximized. The firm equilibrium can be explained with the help of two approaches.

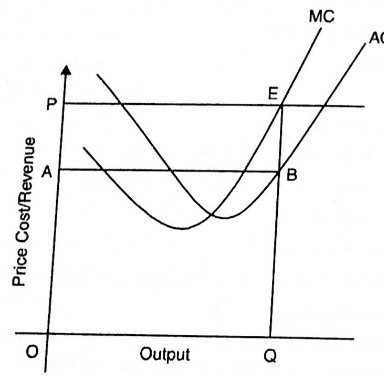
1. Marginal Revenue and Marginal Cost approach
2. Total Revenue and Total Cost Approach.

According to MR and MC approach a firm is said to be in equilibrium if the following conditions are fulfilled.

(i) Marginal Cost (MC) is equal to Marginal Revenue (MR) i.e. MC = MR.

(ii) Marginal Cost (MC) cuts marginal Revenue (MR) from below.

(iii) Marginal Cost (MC) cuts average cost (AC) from the Minimum point.



Here, MR and MC intersect each other at a point E, which is known as equilibrium point. OQ is the equilibrium output and profit maximizing output is also OQ. The firm can maximize its profit upto MR when greater than MC i.e. MR > MC. Profit is possible until MR is equal to MC i.e. MR = MC.

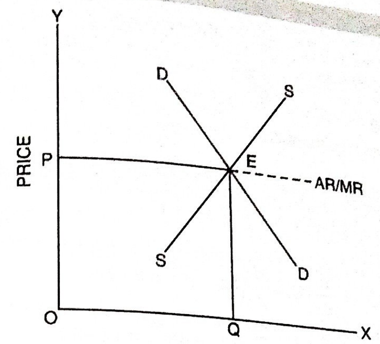
2. If MC > MR then firm level of output bear losses. Similarly, industry is the group of firm producing homogeneous product or commodity. In the perfect competition market, the equilibrium in industry must fulfill following conditions.

(i) Quantity demanded be equal to quantity supply i.e. QD = QS

(ii) Marginal cost (MC) be equal to Marginal Revenue i.e. MC = MR.

(iii) Marginal cost (MC) cuts Marginal Revenue form below.

Here, supply curve (S) and Demand Curve (D) and, AR and MR are equal. An industry is in equilibrium at the point E. OP is the equilibrium price and OQ is the equilibrium output



Mathematically,

When, Profit (P) = R –C, R = Revenue, C = Cost and R = $f(x)$ C = $f(x)$ P = $f(x)$

In order to find maximization of profit

Step I. Find $\frac{dP}{dx}$, when P = R –C

Step II.

$$\frac{dP}{dx} = \frac{d}{dx}(R-C) = \frac{d}{dx}(R) - \frac{d}{dx}(C)$$

$$\frac{dP}{dx} = MR - MC$$

As a necessary condition Marginal Profit (MP) will be zero in a equilibrium point.

Thus, $\frac{dP}{dx}$ = MP = 0 and again MR – MC = 0

Step III. Again, $\frac{dP}{dx}$ = MR –MC or $\frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(MR) - \frac{d}{dx}(MC)$

$$\frac{d^2}{dx^2}\left(\frac{dP}{dx}\right) = \frac{d(MR)}{dx} - \frac{d}{dx}(MC)$$

In case of profit maximization $\frac{d^2P}{dx^2} < 0$

$$\frac{d(MR)}{dx} - \frac{d(MC)}{dx} < 0$$

or $\frac{d(MR)}{dx} < \frac{d(MC)}{dx}$ or $\frac{d}{dx}(MC) > \frac{d}{dx}(MR)$

Thus, change in MC > change in MR.

ILLUSTRATION 18. Find out the equilibrium quantity and price in different demand and cost functions of different firms.

(a) $p = 12 - 4x$ and $C(x) = 8x - x^2$

(b) $p = \frac{75-x}{3}$ and $C(x) = 100 + 3x$

(c) $p = 10e^{-x}$ and $C(x) = 10$

In all the case p is price and x is quantity.

SOLUTION

$p = 12 - 4x$ and $C(x) = 8x - x^2$

$\therefore$ R(x) = Price × Quantity $= (12 - 4x) \times x = 12x - 4x^2$

Since, $C(x) = 8x - x^2$

$$MC = \frac{d}{dx}(C) = \frac{d}{dx}\left(8x - x^2\right) = 8 - 2x.$$

$$MR = \frac{d}{dx}(R) = \frac{d}{dx}\left(12x - 4x^2\right) = 12 - 8x$$

In an equilibrium condition MC = MR

Thus, $8 - 2x = 12 - 8x$ or $6x = 4$ or $x = \frac{4}{6} = \frac{2}{3}$

Price $(p) = 12 - 4x = 12 - 4 \times \frac{2}{3} = 12 - \frac{8}{3} = ₹\frac{28}{3}$

Thus, the equilibrium quantity is $\frac{2}{3}$ and price is ₹ $\frac{28}{3}$

(b) Given, $p = \frac{75-x}{3}$ and $C(x) = 10$

$$R(x) = p.x = \left(\frac{75x - x^2}{3}\right) = \frac{d}{dx}\left(25x - \frac{1}{3}x^2\right) = 25 - \frac{2}{3}x$$

Now, MC = MR

Thus, $3 = 25 - \frac{2}{3}x$ or $\frac{2}{3}x = 22$

Or $x = 22 \times \frac{3}{2} = 33.$

$$\text{Price (P)} = \frac{75-x}{3} = \frac{75-33}{3} = \frac{42}{3} = ₹\ 14$$

Thus, the equilibrium quantity is 33 and price is ₹ 14.

(C) Given, $p = 10e^{-x}$ and $C(x) = 10$

$$R(x) = p \times x = 10e^{-x} \times x = 10x.\ e^{-x}$$

$$C(x) = 10$$

$$MC = \frac{d}{dx}(C) = \frac{d}{dx}(10) = 0$$

$$MR = \frac{d}{dx}(R) = \frac{d}{dx}10x.e^{-x}$$

$$= e^{-x}\frac{d}{dx}(10x) + 10x.\frac{d}{dx}e^{-x}$$

$$= e^{-x} \times 10 + 10x(-1)e^{-x} = 10e^{-x} - 10x.e^{-x}$$

$$= 10e^{-x}(1 - x)$$

Now, $1 - x = 0$ or $x = 1$

Price $(p) = 10e^{-x} = 10 \times e^{-1} = 10 \times 0.36788 = ₹\ 3.68$

$\therefore$ Equilibrium Quantity = 1 and Price = ₹ 3.68.

EXERCISE D

1. The total cost of making x units of a commodity is C where. $C = 150 + 5x + 0.01\ x^2$. Find the (*i*) total cost of making 100 units, and 101 units, (ii) marginal cost at 100 and at 101 units. **[Ans.** (*i*) 750, 757.01, (*ii*) 7, 7.02]

2. If the total cost $C = 10 + 30\sqrt{x}$, then find the average cost at 100 gallons of output. Also, find the level of outputs at which the marginal cost is ₹ 0.40 per gallon. **[Ans.** 1.5, $1406\frac{1}{4}$]

3. The total cost of making q units is given by $C = \frac{q^3}{3000} - 0.2\ q^2 + 27.5\ q$, and all the units made are sold at ₹ 10 per unit. At which two points does the marginal cost equal the marginal revenue ? What number of units lead to maximum profit? **[Ans.** 350, 50, 350]

4. Find the elasticity of demand when the demand is given by $q = \frac{20}{p+1}$, and $p = 3$. **[Ans.** 3/4]

5. If the demand law is $q = $ Ae–KP, express the demand elasticity, and total, average and marginal revenue as the functions of q.

   [Ans. $\log A - \log q, \frac{q}{K}(\log A - \log q)$ $(\log A - \log q)\frac{1}{K}(\log A - \log q - 1)$]

6. Find the elasticity of demand w.r.t. price for the following demand functions :

   (*a*) $\sqrt{a - bD}$, a and b being constants; (*b*) $D = \frac{8}{p^{3/2}}$. (*c*) $D = p^a e^{-b(p+C)}$ where a, b, c, are constants. **[Ans.** (*a*) $\frac{2}{bd}(a - bD)$; (*b*) 3/2 (*c*) $(a - bp)$.

7. Show that the elasticity of demand at all the points of the curve $xy = \alpha 2$ will be numerically equal to one.

8. Let the cost function of a firm be given by $C = 300x - 10x^2 + \frac{1}{3}x^3$, where C stands for the cost, and x for the output, Find the output at which (*i*) the MC is the minimum (*ii*) the AC is the minimum and (*iii*) the AC is equal to the MC. **[Ans.** (*i*) at 10, (*ii*) at 15, (*iii*) at 15]

9. The total revenue function of a firm is given by $R = 21q - q^2$, and its total cost function by

$$C = \frac{1}{3}q^3 - 3q^2 - 7q + 16 \text{ where, } q \text{ is the output.}$$

Determine the output at which (*i*) the total revenue is maximum, and (*ii*) the total cost is minimum. [**Ans.** 10.5, 7]

10. If $x = 25 - 3p - p^2$ be a demand function, then find the point elasticity of demand at $p = 3$. [**Ans.** 27/7]

11. For the demand curve $aQ + bP - K = 0$, where a, b and K are +ve constants. Determine the point elasticity of demand when the marginal revenue is zero. [**Ans.** 1]

12. Verify that $\eta_d = \dfrac{AR}{AR-MR}$ for the linear demand law $p = a + bx$. [**Ans.** $\dfrac{a+b}{-bx} = \eta_d$ proved]

13. What is the marginal revenue for a demand curve which has infinite elasticity? [**Ans.** MR = AR]

14. Find the elasticity of total cost, and average cost of the function $C = 2x^2 + 4x + 3$

[**Ans.** (*i*) $\dfrac{x(4x+4)}{2x^2+4x+3}$ (*ii*) $\dfrac{2x^2-3}{2x^2+4x+3}$]

15. Determine the equilibrium price given, $q_d = \dfrac{8p}{p-2}$, $q_s = p^2$. [**Ans.** $p = 4$]

16. Find the equilibrium point, if the supply and demand functions of a product are:

$P = \dfrac{3x}{100} + 2$ and $p = \dfrac{-7x}{100}100 + 12$ respectively. [**Ans.** (10, 5)]

17. Determine the equilibrium price and quantity from the supply and demand functions as follows:

$P_s = 2x + 20$, $P_d = 200 - 2x^2$ [**Ans.** 9, 3x]

18. Find the equilibrium price and quantity for the two commodity market models as follows:

$$Q_{d1} = -2 - p + q\,; \qquad Q_{s1} = -2 - q$$
$$Q_{d2} = -3 - p - q\,; \qquad Q_{s2} = -9 + pq$$

where p is price, and q is quantity.

19. Find the market equilibrium price, and quantity, if the demand laws for two commodities are as follows : $p = 24 - x - 2y$, $q = -27 - x - 3y$

And supply laws are : $x = -6 + 2p - q$; $y = -3 - p - 8q$ where p and q represent the price per unit of commodity, x and y respectively.

20. The demand and supply relations are given by the equations : $p^2 + q^2 = 20$, and $2p + q = 8$ respectively, where p is the price and q is the quantity. Find the equilibrium price and quantity. [**Ans.** 2, 4]

10. PARTIAL DERIVATIVE IN ECONOMICS

If L units of labour and K units of capital are used to produce **P** units of a product, then the function **P** = *f*(L, K) is called **production function** and its partial derivatives are called **marginal productivity functions.** The concept of partial derivatives are also used to find the nature of commodities. Again, it is very much useful to solve applied extreme problems. Suppose, a monopolist producing two products with known market demands might be interested to know about the quantity of items of each be produced to have the joint profit maximum. The partial derivative can be used as a tool to find the answers of economic problems.

(A) DEMAND ANALYSIS

Let X_1 and X_2 be two related Commodities, then the demand for each product is dependent on the prices of both. Suppose, x_1 and x_2 are the quantities demanded for X_1 and X_2 respectively, and p_1 and p_2 are their respective prices. Then both x_1 and x_2 are functions of p_1 and p_2. Thus,

$x_1 = f(p_1, p_2)$, demand function of X_1

$x_2 = g(p_1, p_2)$, demand function of X_2.

Generally, demand function for each commodity is assumed to be continuous.

So, the **partial marginal demand functions are :**

(i) The partial marginal demand for X_1 with respect to p_1 is defined as $\dfrac{\partial x_1}{\partial p_1}$. This represents the rate of change of x_1 with respect to p_1 and p_2 is held fixed.

(ii) The particular marginal demand for X_1 with respect to p_2 is defined as $\dfrac{\partial x_1}{\partial p_2}$ is defined as $\dfrac{\partial x_1}{\partial p_2}$. This is the rate of change of x_1, with respect to p_2 when p_1 is held fixed.

(iii) The partial marginal demand for X_2 with respect to p_1 is defined as $\dfrac{\partial x_2}{\partial p_1}$. This gives the rate of change of x_2 with respect to p_1 when p_2 is held fixed.

(iv) The partial marginal demand for X_2 with respect to p_2 is defined as $\dfrac{\partial x_2}{\partial p_2}$. This gives the rate of change of x_2 with respect to p_2 when p_1 is held fixed.

Note : In a normal condition, if the price of X_2 is fixed and the price of X_1 increases then the quantity of X_1 demanded will decrease. Hence, $\dfrac{\partial x_1}{\partial p_1} < 0$ and similarly, $\dfrac{\partial x_1}{\partial p_2} < 0$. However $\dfrac{\partial x_1}{\partial p_2}$ and $\dfrac{\partial x_2}{\partial p_1}$ may be either positive or negative.

ILLUSTRATION 19. The demand functions of two commodities X and Y are given by,

$$x = 6q^2 - 2pq \text{ and } y = 9p^2 - pq$$

Where, x and y are quantity demanded for X and Y respectively and p and q are their respective prices. Find, the four partial marginal demands when $p = 100$ and $q = 140$. Interpret the result.

SOLUTION : The partial marginal demand for X with respect to

$$p = \frac{\partial x}{\partial p} = \frac{\partial}{\partial p}(6q^2 - 2pq) = 0 - 2p = -2p$$

Partial marginal demand for X with respect to $q = \dfrac{\partial x}{\partial q} = \dfrac{\partial}{\partial q}(6q^2 - 2pq) = 12q - 2p$

When, $p = 100$ and $q = 140$, we get

$$\frac{\partial x}{\partial p} = -2q = -2 \times 140 = -280$$

and $\dfrac{\partial x}{\partial q} = 12q - 2p = 12 \times 140 - 2 \times 100 = 1680 - 200 = 1480$

Thus, if $p = 100$ and $q = 140$, increasing p to 101 and holding q at 140 will decrease the demand by 280 units. But if q is increased to 141 while p is held at 100, the demand for X increases by 1480 units.

Further, Partial Marginal demand for y with respect to $p = \frac{\partial y}{\partial p} = \frac{\partial}{\partial p}(qp^2 - pq) = 18p - q.$

Partial Marginal demand for y with respect to q.

$$= \frac{\partial y}{\partial q} = \frac{\partial}{\partial q}(qp^2 - pq) = -p$$

When, $p = 100$ and $q = 140$, we get

$$\frac{\partial y}{\partial p} = 18p - q = 18 \times 100 - 140 = 1660$$

$$\frac{\partial y}{\partial q} = -p = -100$$

Thus, if $p = 100$ and $q = 140$, increasing p to 101 and holding q at 140 will increase in demand for Y by 1660 units. But if q increased to 141 while p held at 100, the demand for X decreased by 100 units.

(B) NATURE OF COMMODITIES

Two commodities are said to be complementary if an increase of price of one of them causes a decrease in the demand for the other, provided the price of the other commodities is held fixed. For example, car and petrol are complementary commodities. The two commodities X_1 and X_2 are **complementary** if and only if $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are both negative.

Depending on the different situation, two commodities are said to be competitive or substitutes if an increase in the price of one of them causes are increase in the demand for the other, provided the price of the other commodity is held fixed. For example, Tea and Coffee are substitutes.

Thus, two commodities X_1 and X_2 are **competitive or substitutes** if and only if $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are both positive.

ILLUSTRATION 20. The demand functions for two commodities X_1 and X_2 are given by

$$x_1 = \frac{200}{p_1^2 p_2} \text{ and } x_2 = \frac{100}{p_1 p_2^2}$$

Find the four partial marginal demand function. Also determine whether X_1 and X_2 are complementary or competitive.

SOLUTION : The partial marginal demand for X_1 with respect to p_1

$$= \frac{\partial x_1}{\partial p_1} = \frac{\partial}{\partial p_1}\left(\frac{200}{p_1^2 p_2}\right) = \frac{\partial}{\partial p_1} 200 p_1^{-2} p_2^{-1} = -\frac{400}{p^3 p_2}$$

The partial marginal demand for X_1 with respect to p_2

$$= \frac{\partial x_1}{\partial p_2} = \frac{\partial}{\partial p_2}\left(\frac{200}{p_1^2 p_2}\right) = -\frac{200}{p_1^2 p_2^2}$$

The partial marginal demand for X_2 w.r.t. p_1

$$= \frac{\partial x_2}{\partial p_1} = \frac{\partial}{\partial p_1}\left(\frac{100}{p_1 p_2^2}\right) = -\frac{100}{p_1^2 p_2^2}$$

The partial marginal demand for X_2 w.r.t. p_2

$$= \frac{\partial x_2}{\partial p_2} = \frac{\partial}{\partial p_2}\left(\frac{100}{p_1 p_2^2}\right) = -\left(\frac{200}{p_1 p_2^3}\right)$$

Since, p_1 and p_2 represent prices and they are both positive. Hence $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are both negative.

Thus, X_1 and X_2 are complementary commodity.

Partial Elasticity of Demand

The partial elasticity of demand is the ratio of the proportional change in quantity demanded of one commodity to the proportional change in price of one commodity (x_1 or x_2), with the price of other commodity (x_2 or x_1) held constant.

(i) The (direct) partial elasticity of demand for X_1 with respect to p_1, denoted by η_{11}(eta), is defined as $\eta_{11} = \frac{p_1}{x_1} \cdot \frac{\partial x_1}{\partial p_1} = \frac{\partial(\log x_1)}{\partial(\log p_1)}$

(ii) The (cross) partial elasticity of demand for X_1 with respect to p_2, denoted by η_{12}, is defined as $\eta_{12} = \frac{p_2}{x_1} \cdot \frac{\partial x_1}{\partial p_2} = \frac{\partial(\log x_1)}{\partial(\log p_2)}$

(iii) The (cross) partial elasticity of demand for X_2 with respect to p_1, denoted by η_{21}, is defined as $\eta_{21} = \frac{p_1}{x_2} \cdot \frac{\partial x_2}{\partial p_1} = \frac{\partial(\log x_2)}{\partial(\log p_1)}$

(iv) The (direct) partial elasticity of demand for X_2 with respect to p_2 denoted η_{22}, is defined as $\eta_{22} = \frac{p_2}{x_2} \cdot \frac{\partial x_2}{\partial p_2} = \frac{\partial(\log x_2)}{\partial(\log p_2)}$

The η_{11} (and η_{22}) can be interpreted as the ratio of a percentage change in the quantity of X_1 (respectively X_2) demanded to a percentage change in the price of X_1 (respectively X_2) when price of X_2 (respectively X_1) is fixed. Similarly, η_{12} and η_{21} can be interpreted.

If η_{12} and η_{21} are both positive, the commodities are competitive but if the both are negative, commodities are **Complementary**. These η_{12} and η_{21} are called **Cross Partial elasticity of demand** and, η_{11} and η_{22} are called **Direct partial elasticity of demand.**

ILLUSTRATION 21. The demand functions of two commodities, X_1 and X_2, are, $x_1 = p_1^{-1.4} p_2^{0.6}$ and $x_2 = p_1^{0.5} \cdot p_2^{-1.2}$ respectively. Where, x_1 and x_2 are the quantity demanded of X_1 and X_2 respectively, and p_1 and p_2 are their respective prices. Find the four partial elasticities of demand and determine whether the commodities are competitive or complementary.

SOLUTION : We have, $\frac{\partial x_1}{\partial p_1} = \frac{\partial}{\partial p_1}(p_1^{-1.4} p_2^{0.6}) = -1.4 p_1^{-2.4} \cdot p_2^{0.6}$

$$\frac{\partial x_1}{\partial p_2}=\frac{\partial}{\partial p_2}(p_1^{-1.4}p_2^{0.6})=0.6p^{-1.4}.p_2^{-0.4}$$

$$\frac{\partial x_2}{\partial p_1}=\frac{\partial}{\partial p_1}(p_1^{0.5}p_2^{-1.2})=0.5p^{-0.5}.p_2^{-1.2}$$

$$\frac{\partial x_2}{\partial p_2}=\frac{\partial}{\partial p_2}(p_1^{0.5}p_2^{-1.2})=-1.2p_1^{0.5}.p_2^{-2.2}$$

The (direct) partial elasticity of demand of X, w.r.t. p_1 is

$$\eta_{11}=\frac{p_1}{x_1}.\frac{\partial x_1}{\partial p_1}=\frac{p_1}{p_1^{-1.4}p_2^{0.6}}\times -1.4p_1^{-2.4}.p_2^{0.6}=-1.4$$

The (cross) partial elasticity of demand of X_1 w.r.t.p_2 is

$$\eta_{12}=\frac{p_2}{x_1}\times\frac{\partial x_1}{\partial p_2}=\frac{p_2}{p^{-1.4}p_2^{0.6}}\times 0.6p_1^{-1.4}p_2^{0.4}=0.6$$

The (cross) partial elasticity of demand of X_2 w.r.t. p_1 is

$$\eta_{21}=\frac{p_1}{x_2}\times\frac{\partial x_2}{\partial p_1}=\frac{p_1}{p_1^{0.5}p_2^{-1.2}}\times 0.5p_1^{-0.5}p_2^{-1.2}=0.5$$

The (direct) partial elasticity of demand X_2 with respect to p_2 is

$$\eta_{22}=\frac{p_2}{x_2}\times\frac{\partial x_2}{\partial p_2}=\frac{p_2}{p_1^{0.5}p_2^{-1.2}}\times -1.2p_1^{0.5}p_2^{2.2}=-1.2$$

The value of both η_{12} and η_{21} are positive and it reveals that X_1 and X_2 are competitive.

ILLUSTRATION 22. The demand function for two commodities are given as $x_1=\frac{-p_1}{p_2^2}$ and $x_2=\frac{p_1^2}{p_2}$ where, p_1 and p_2 are prices and x_1 and x_2 denote the quantities of the two commodities respectively. Show that the two commodities are substitutes for one another.

SOLUTION : The four marginal demands are given by

$$\frac{\partial x_1}{\partial p_1}=\frac{-1}{p_2^2},\ \frac{\partial x_1}{\partial p_2}=-p_1(-2)p_2^{-3}=\frac{2p_1}{p_2^3}$$

$$\frac{\partial x_2}{\partial p_1}=\frac{1}{p_2}\times 2p_1=\frac{2p_1}{p_2}$$

$$\frac{\partial x_2}{\partial p_2}=p_1^2(-1).p_2^{-2}=-\frac{p_1^2}{p_2^2}=-\left(\frac{p_1}{p_2}\right)^2$$

Now, $\frac{\partial x_1}{\partial p_1}$ and $\frac{\partial x_2}{\partial p_2}$ are negative and $\frac{\partial x_2}{\partial p_1}$ and $\frac{\partial x_1}{\partial p_2}$ are positive. Hence the commodities are substitutes.

Note : (i) Two commodities are complementary if both $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are negative

(ii) Two commodities are substitutes (or competitive) if both $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are positive.

ILLUSTRATION 23. Find the partial elasticities $z=x^2e^y$

SOLUTION : Partial elasticity of z w.r.t. x is

$$\eta_{z,x}=\frac{x}{z}\frac{\partial z}{\partial x}=\frac{x}{x^2e^y}2x\,e^y=2$$

Partial elasticity of z w.r.t y is

$$\eta_{z,y}=\frac{y}{z}\frac{\partial z}{\partial y}=\frac{y}{x^2e^y}x^2.e^y=y$$

ILLUSTRATION 24. If the supply function $x=f(p_1,p_2,\ldots p_m)$ is homogeneous of degree n, **show the sum of the partial price elasticities of supply equals n.** (x denotes the quantity supplied of a particular commodity and $p_1,p_2,\ldots p_m$ are the prices of different commodities.

SOLUTION : Let η_r = partial price elasticity of supply for X w.r.t. p_r $(r=1,2,\ldots m)$, then

$$\eta_r=\frac{p_r}{x_r}\frac{\partial x}{\partial p_r}$$

Since, $x=f(p_1,p_2,\ldots p_m)$is homogeneous of degree n, according to Euler's theorem,

$$p_1\frac{\partial x}{\partial p_1}+p_2\frac{\partial x}{\partial p_2}+\ldots+p_m\frac{\partial x}{\partial p_m}=nx$$

Dividing both sides of this equation by x, we get

$$\frac{p_1}{x}\frac{\partial x}{\partial p_1}+\frac{p_2}{x}\frac{\partial x}{\partial p_2}+\ldots+\frac{p_m}{x}.\frac{\partial x}{\partial p_m}=n$$

i.e. $\eta_1+\eta_2+\ldots+\eta_m=n$

Thus, the sum of partial price elasticities of supply equals n.

ILLUSTRATION 25. The demand functions for two commodities X_1 and X_2 in terms of their respective prices p_1 and p_2 are given by

$$x_1=p_1^{-a_1}.e^{b_1p_2+c_1}\text{ and }x_2=p_2^{-a_2}.e^{b_2p_1+c_2}$$

where a_1, a_2, b_1, b_2 and c_1, c_2 are constants

(i) Find the four partial marginal demand functions.

(ii) Determine the conditions so that the commodities are (a) and (b) complementary.

(iii) Show that the direct partial elasticities of demand are determined in sign by the constants b_1 and b_2 respectively.

SOLUTION :

(i) The partial marginal demand of X_1 with respect to $p_1=\frac{\partial x_1}{\partial p_1}=-a_1p_1^{-a_1-1}.e^{b_1p_2+c}=-\frac{a_1x_1}{p_1}$

The partial marginal demand of X_1 with respect to $p_2 = \frac{\partial x_1}{\partial p_2} = b_1 p_1^{-a_1} . e^{b_1 p_2 + c_1} = b_1 x_1$

The partial marginal demand of X_2 with respect to $p_1 = \frac{\partial x_2}{\partial p_1} = b_2 p_2^{-a_2} . e^{b_2 p_1 + c_2} = b_2 x_2$

The partial marginal demand of X_2 w.r.t. $p_2 = \frac{\partial x_2}{\partial p_2} = -a_2 p_2^{-a_2 - 1} . e^{b_2 p_1 + c_2} = -\frac{a_2 x_2}{p_2}$

(ii) The commodities X_1 and X_2 will be competitive or complementary depending upon whether $\frac{\partial x_1}{\partial p_2}, \frac{\partial x_2}{\partial p_1}$ are both positive or negative. Here, $\frac{\partial x_1}{\partial p_2} = b_1 x_1$ and $\frac{\partial x_2}{\partial p_1} = b_2 x_2$

Since, x_1 and x_2 are both positive, the signs of $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ depend on the signs of b_1 and b_2 respectively. Therefore, X_1 and X_2 will be competitive if both b_1 and b_2 are positive and complementary if both b_1 and b_2 are negative.

(iii) The two direct partial elasticities of demand are,

$$\eta_{11} = \frac{p_1}{x_1} . \frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\left(-a_1 \frac{x_1}{p_1}\right) = -a_1$$

and
$$\eta_{22} = \frac{p_2}{x_2} . \frac{\partial x_2}{\partial p_2} = \frac{p_2}{x_2}\left(-a_2 \frac{x_2}{p_2}\right) = -a_2$$

Since, a_1 and a_2 are constants, it follows that η_{11} and η_{22} are independent of prices. The two cross partial elasticities of demand are :

$$\eta_{12} = \frac{p_2}{x_1} . \frac{\partial x_1}{\partial p_2} = \frac{p_2}{x_1}(b_1 x_1) = b_1 b_2$$

and
$$\eta_{21} = \frac{p_1}{x_2} . \frac{\partial x_2}{\partial p_1} = \frac{p_1}{x_2}(b_2 x_2) = b_2 b_1$$

Since, p_1 and p_2 are both positive , it reveals that the η_{12} and η_{21} are determined in sign by the constants b_1 and b_2 respectively.

Note : (1) The concept of partial elasticities can be extended to demand functions of more than two variables. Suppose, if demand for any commodity X_r is a function of all market prices.

$x_r = f_r(p_1, p_2, p_3 ... p_n)$ $(r = 1, 2, 3, ...n)$ then the partial elasticity of X_r w.r.t. p_s, denoted by η_{rs} is given by

$$\eta_{rs} = \frac{p_s}{x_r} = \frac{\partial x_r}{\partial p_s} = \frac{\partial(\log x_r)}{\partial(\log p_s)}$$

(2) The partial elasticities of demand can be extended to arbitrary functions of n variables

For example : $z = f(x_1, x_2, x_3, ...x_n)$, the partial elasticity of Z w.r.t. x_r is defined to be $\frac{x_r}{z} . \frac{\partial z}{\partial x_r} = \frac{\partial(\log z)}{\partial(\log x_r)}$. In particular, if the supply function for a commodity X_r is given by

$$x_r = f_r(p_1, p_2, p_3 ..., p_n)$$

Where x_r is the quantity of X supplied when the price per unit X_1 is p_1, the price per unit X_2 is p_2 and so on, then the partial elasticity of supply of X_r with respect to p_s is resulted by $\frac{p_s}{x_r} \times \frac{\partial x_r}{\partial p_s}$.

ILLUSTRATION 26. The demand function for mutton is $Q_M = 4850 - 5P_M + 1.5\,P_O + 0.1\,Y$

Find (i) Income elasticity of demand

(ii) Cross elasticity of demand for mutton at Y (income) = ₹ 1,000

P_M (Price of Mutton) = ₹ 200

P_O (Price of Chicken) = ₹ 100

SOLUTION : Income elasticity $= \frac{dq}{dy} \times \frac{Y}{Q}$

$$\frac{dq}{dy} = 0.1,\ Y = 1000$$

$$Q = 4{,}850 - 5 \times 200 + 1.5 \times 100 + 0.1 \times 1{,}000$$
$$= 4{,}850 - 1{,}000 + 150 + 100 = 4{,}100$$

(i) Income elasticity $= \frac{0.1 \times 1{,}000}{4{,}100} = \frac{1}{41}$ **Ans.**

(ii) Cross elasticity of demand for Motton :

$$\frac{p_o}{q_m} \times \frac{\partial q_m}{\partial p_o} = \frac{100}{4{,}100} \times 1.5 = \frac{15}{410} \text{ or } \frac{3}{82}$$

ILLUSTRATION 27. Minimize Costs for a firm with the cost function $C = 5x^2 + 2xy + 3y^2 + 800$. Subject to the production quota $x + y = 39$ and estimate the additional costs if the production quota is increased by 40.

SOLUTION : $L = 5x^2 + 2xy + 3y^2 + 800 + \lambda\,(3.9 - x - y)$ and $\frac{\partial L}{\partial x} = 10x + 2j - \lambda = 0$

$$\frac{\partial L}{\partial y} = 2x + 6y - \lambda = 0 \qquad ....(2)$$

$$\frac{\partial L}{\partial \lambda} = 39 - y - y = 0 \qquad ...(3)$$

When solved simultaneously, the equations (1) and (2) we get $x = 13$ and $y = 26$.

$$\lambda = 10x + 2y = 130 + 52 = 182$$

Now minimum Cost $= 5x^2 + 2xy + 3y^2 + 800$

$$= 5 \times (13)^2 + 2 \times 13 \times 26 + 3(26)^2 + 800 = 4339$$

Here, when the production quota is increased by one unit is from 39 to 49, the additional cost value of $\lambda = 182$.

EXERCISE E

1. The demand functions for two commodities X and Y are $x = \frac{10}{p_1^2 p_2}$ and $y = \frac{5}{p_1 p_2^2}$ respectively,

where p_1 and p_2 are the prices of the commodities. Find $\frac{\partial x}{\partial p_1}, \frac{\partial x}{\partial p_2}, \frac{\partial y}{\partial p_1}$ and $\frac{\partial y}{\partial p_2}$ interpret them in economic sense. Examine whether the two commodities are complementary or competitive.

[**Ans.** $\frac{\partial x}{\partial p_1} = \frac{-20}{p_1^3 p_2}, \frac{\partial x}{\partial p_2} = \frac{-10}{p_1^2 p_2} \frac{\partial y}{\partial p_1} = \frac{-5}{p_1^2 p_2^2}, \frac{\partial y}{\partial p_2} = \frac{10}{p_1 p_2^3}$]

2. Discuss whether the two goods are substitutes or complementary on the basis of demand functions as given below : $x_1 = p_1^{-.4} e^{.2p_2}, x_2 = p_2^{-.6} . e^{5p_1}$, where p_1 and p_2 denote the prices and x_1, x_2 are the quantities of the two goods respectively.

(**Ans.** Substitutes)

3. The demand functions for two commodities are given below :

$$x_1 = -\frac{p_1}{p_2^2}, x_2 = \frac{p_1^2}{p_2}$$

where p_1 and p_2 are prices and x_1, x_2 denote the quantities of the twp commodities respectively. Show that the two commodities are substitutes for one another.

4. The demand functions for commodities X_1 and X_2 are each a function of the prices of X_1 and X_2 and are given by $x_1 = -\frac{4}{p_1^2 p_2}$ and $x_2 = \frac{16}{p_1 p_2^2}$, respectively. Find the four partial marginal demand functions and determine whether X_1 and X_2 are competitive, complementary or neither. Also determine the four partial elasticities of demand.

5. If the demand laws of two goods are given as :

$$x_1 = p_1^{-a_{11}} e^{a_{12} p_2 + a_1} \text{ and } x_2 = p_2^{-a_{22}} e^{a_{21} p_1 + a_2}$$

Find the condition that goods are (i) competitive and (ii) complementary.

[**Ans.** (i) Competitive if both $a11$ and $a21$ are positive (ii) Complementary if both are negative.]

6. For the demand functions of two commodities given below, find the four partial elasticities of demand with respect to price and indicate whether the commodities are competitive or complementary :

$$x_1 = 2p_1^{-0.6} p_2^{0.8} \qquad x_2 = 3p_1^{0.7} p_2^{-0.5}$$

(where x_1 and x_2 are the quantities demanded of the two commodities at prices p_1 and p_2 respectively).

[**Ans.** $\eta_{11} = -0.6, \eta_{21} = 0.8, \eta_{21} = 0.7, \eta_{22} = -0.5$, Competitive]

7. If x_1 and p_1 are the demand and price of tea, and x_2 and p_2 are demand and price of coffee and the demand functions are given by

$$x_1 = p_1^{-1.7} p_2^{0.6} \quad \text{and } x_2 = p_1^{0.4} p_2^{-0.8},$$

calculate the two cross elasticities of demand and point out whether the commodities are competitive or complementary.

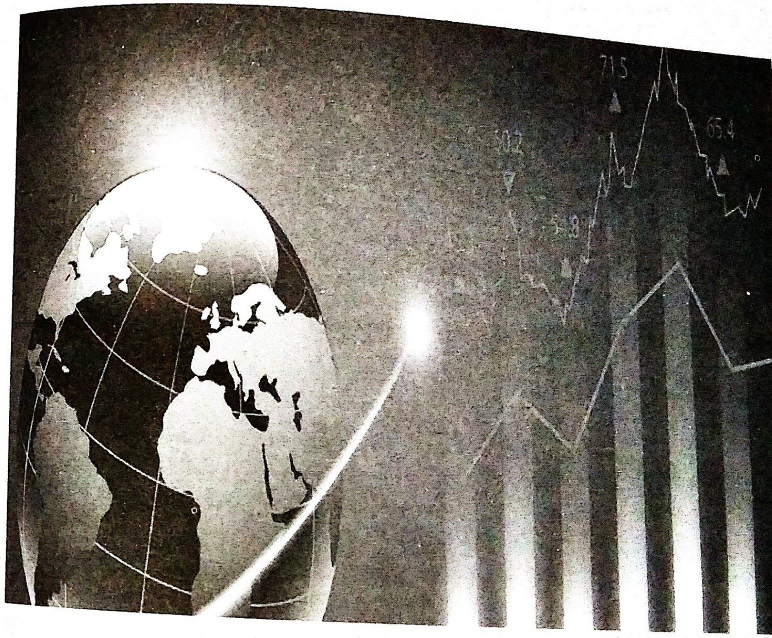
[**Ans.** $\eta_{12} = 0.6$ competitive]

□□□

Unit – IV

Matrics and Determinants (With Application in Economic Models)

7. Matrices and Determinants



Chapter 7

MATRICES AND DETERMINANTS

1. DEFINITION

A matrix may be defined as an orderly arrangement of some numbers and symbols in certain rows and columns enclosed by some brackets, subscripted by the magnitude of its order and denominated by some capital letter.

The following are the specimens of a matrix :

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ (ii) $B = \begin{bmatrix} 15 & 18 \\ 20 & 15 \\ 30 & 40 \end{bmatrix}_{3\times2}$

(iii) $C = \begin{pmatrix} x & y & z \\ p & q & r \end{pmatrix}_{2\times3}$

2. ESSENTIAL CHARACTERISTICS

From the above definition and the specimens, the essential characteristics of a matrix may be analysed as under :

(i) It consists of some numbers or symbols. The numbers like, 0, 5, 10, 125, 3500, and the symbols like x, y, z etc. constitute a matrix. These are called the elements of a matrix without which a matrix cannot come into existence. These numbers may take any sign and any form like, ± integers, ± decimals like 0.35, –0.75, ± fractions like 3/7, –2/11, –7/9 and ± mixed numbers like, 10.75, –3.375 etc. They may consist of single digits or multiple digits including only zeroes even. However in order to constitute a matrix, they must be orderly arranged in some rows and columns. **Any disorderly scattered numbers, or symbols will not constitute a matrix.**

For example, the following groups of numbers and symbols will not amount to matrices :

$$\text{(a)} \begin{pmatrix} & 2 & & & 3 & \\ 1 & & 6 & 5 & & 4 \\ & 8 & & 7 & 9 & \end{pmatrix} \; ; \; \text{(b)} \begin{pmatrix} & P & \\ y & x & z \\ & n & m \end{pmatrix}$$

Further, the elements should be so arranged that each of them is capable of being subscripted by its ith row and jth column to locate its position in the matrix. Thus, if an element say, 15 is subscripted as $15_{1,2}$, it would indicate that the said element 15 lies in the first row and second column of the matrix. In a

disorderly arrangement of numbers like the above ones a number cannot be subscripted by its ith row and jth column.

***(ii)* It consists of some rows and columns.** A matrix always consists of certain rows and columns in which all its elements are arranged. The number of such rows and columns may be one or more and there may or may not be equality between the number of rows and the number of columns. But a column or a row must be complete with some elements.

Thus a group of rows and columns not completed with all its elements as follows will not amount to a matrix :

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 5 \\ 2 & \\ 3 & 4 \end{pmatrix}$

It may be noted that an empty space in a row or a column is not equal to '0' for that a '0' is an element whereas an empty space is never an element of a matrix.

***(iii)* It must be enclosed by some brackets.** A group of orderly arranged numbers or symbols to be called a matrix must be enclosed by some brackets viz. parentheses ( ), square brackets [ ], or curly brackets { }.However, conventionally, the curly brackets are not used in representing a matrix.

Thus, a group of following numbers and symbols not encompassed by any bracket will not constitute a matrix :

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 5 \\ 2 & \\ 3 & 4 \end{pmatrix}$

(iv) It must be subscripted by the magnitude of its order. The magnitude of the order of a matrix refers to the number of rows and columns with which a matrix is constituted. The number of rows and columns must be subscripted at the end and at the bottom of the right hand side bracket of a matrix as $m \times n$ (read as m by n), where m, represents the number of rows and n the number of columns in the matrix. Thus, in a matrix, if the subscript stands like, 4×3, it will mean that there are 4 rows and 3 columns in the said matrix.

(v) It must be denominated by some capital letter. Every matrix must be denominated properly for making a reference to it in the course of computational works. Conventionally, all the matrices are denominated or named by some letters of upper case viz. A, B, C, D etc. Without the proper denomination, any orderly arrangement of numbers or symbols will not constitute a matrix.

Having thus, analysed the whole corpus of a matrix may be represented as under :

$$A = \begin{pmatrix} e_{ij} & e_{ij} & e_{ij} \\ e_{ij} & e_{ij} & e_{ij} \\ e_{ij} & e_{ij} & e_{ij} \end{pmatrix}_{m \times n}$$

where, A refers to the name of the matrix, e to the element or entry in the matrix ; *i.j.* to the subscript of an element in which *i*, indicates the row and *j* the column of the matrix in which the element appears: $m \times n$, to the subscript of the matrix in which m indicates the number of rows and n the number of columns contained in the matrix and ( ), to the enclosure or boundary of the matrix.

Besides, the horizontal lines and the vertical lines in which the elements stand orderly placed are respectively called as the rows and columns of the matrix.

3. DIFFERENT TYPES OF MATRIX

Before entering upon the arithmetic operations on the matrices, it is highly necessary to have an idea about the various types or forms of matrix.

These are identified here as under :

(i) Row Matrix. A matrix that appears with one row only is called a row matrix.

Examples : (i) $A = (0\ 1\ 2)_{1\times3}$; $B = [12\ 22\ 10\ 15]_{1\times4}$

(ii) Column Matrix. A matrix that appears with one column only is called a column matrix.

Examples : (i) $A = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{3\times1}$; (ii) $B = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}_{4\times1}$; (iii) $C = \begin{bmatrix} 15 \\ 25 \\ 35 \\ 45 \\ 40 \end{bmatrix}_{5\times1}$

(iii) Zero or Null Matrix. A matrix that consists of zeroes only is called a zero or null matrix. This is usually denoted by the capital letter O.

Examples :

(i) $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{2\times1}$; (ii) $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$; (iii) $O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{3\times4}$

(iv) Singleton Matrix. A matrix that Comprises one element only is called a singleton matrix.

Examples : (i) $A = [0]_{1\times1}$, (ii) $B = [5]_{1\times1}$ (iii) $C = [25]_{1\times1}$

(v) Square matrix. A matrix that appears with equal number of rows and columns (*i.e.* $m = n$) is called a square matrix.

Examples : (i) $A = (1)_{1\times1}$, (ii) $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times2}$, (iii) $C = \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix}_{3\times3}$

(vi) Diagonal matrix. A square matrix in which all the principal diagonal elements are non-zeroes and all other elements are zeroes is called a diagonal matrix.

Note. (i) *Principal diagonal element.* An element, both the subscripts (*i* and *j*) of which are equal is called a principal diagonal element. The line along with the principal diagonal elements are positioned is called the principal or leading diagonal.

Examples : a_{11}, a_{22}, a_{33}, a_{44} and the like.

(ii) *TRACE.* The sum of the Principal diagonal elements of a square matrix is called **Trace.** In the example (ii), the trace is $1 + 2 + 3 = 6$, similarly in matrix A, the trace is $2 + 5 = 7$.

(vii) Scalar matrix. A diagonal matrix in which all the leading diagonal elements are equal is called a Scalar Matrix. In other words, a square matrix in which all the elements except those in the main diagonal are zeroes and all the leading diagonal elements are equal is called a scalar matrix.

Examples : (i) $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}_{2\times2}$, (ii) $B = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}_{3\times3}$

(viii) Unity or Identity matrix. A square matrix in which all the leading diagonal elements are unity or 1 and all other elements are zeroes is called a unity or identity matrix. It is conventionally denoted by the capital letter, I.

Examples : (i) $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}_{2\times2}$, (ii) $I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}_{3\times3}$, (iii) $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}_{4\times4}$

(ix) Triangular matrix. A square matrix in which all the elements above or below the principal diagonal are zeroes, and the rest are non-zeroes is called a triangular matrix. If the zero elements lie below the principal diagonal, it is called an upper-triangular matrix, and if the zero elements lie above the principal diagonal it is called a lower triangular matrix.

Examples :

(i) $A=\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}_{3\times3}$, (ii) $B=\begin{bmatrix}1&0&0&0\\5&2&0&0\\7&8&3&0\\9&4&6&4\end{bmatrix}_{4\times4}$

An upper triangular matrix *A lower triangular matrix*

(x) Equal matrix. A matrix is said to be equal to another matrix, if all its elements are equal to the corresponding elements of the said another matrix.

Examples :

(i) If $A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}_{2\times3}$; and $B=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}_{2\times3}$, then A=B.

$\therefore$ A is an equal matrix to B and vice versa.

(ii) If $C=\begin{pmatrix}1&7\\4&3\end{pmatrix}_{2\times2}$, and $D=\begin{pmatrix}x&x+y\\4&3\end{pmatrix}_{2\times2}$, where, $x=1$, and $y=6$, then C = D

$\therefore$ C is an equal matrix to D and vice versa.

(xi) Comparable or Equivalent matrix. A matrix is said to be comparable or equivalent to another matrix, if the number of its rows and columns is equal to those of the other matrix, *i.e.* $m_1=m_2$, and $n_1=n_2$

Examples : If $A=\begin{pmatrix}1&7&8\\3&1&2\end{pmatrix}_{2\times3}$, and $B=\begin{pmatrix}15&10&25\\16&8&14\end{pmatrix}_{2\times3}$ then A B.

(xii) Sub matrix. A small matrix obtained by deleting some rows or/ and some columns of a given matrix is called a sub matrix.

If $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, so $S_1=\begin{bmatrix}1&2\\4&5\end{bmatrix}$, $S_2=\begin{bmatrix}2&3\\5&6\end{bmatrix}$, $S_3=\begin{bmatrix}4&5\\7&6\end{bmatrix}$, $S_4=\begin{bmatrix}5&6\\8&9\end{bmatrix}$, $S_5=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$

4. ARITHMETIC OPERATIONS ON MATRICES

The basic arithmetic operations of addition, subtraction, multiplication and division can very well be performed on matrices subject to certain conditions and procedures laid down as under :

(i) Addition of Matrices

Condition Necessary

The matrices to be added to each other must be comparable, *i.e.* each of the matrices must have equal number of rows and equal number of columns. Symbolically, $m_1=m_2=m_3$ and so on, and $n_1=n_2=n_3$ and so on.

Procedure

(a) Place all the matrices to be added in a horizontal line and put + signs between each of the pairs of them.

(b) Add the corresponding elements of each of the matrices and put their sums in the same order.

EXAMPLE 1. Find the sum of addition of the following two matrices :

(i) $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}_{3\times3}$ and $B=\begin{pmatrix}10&11&12\\13&14&15\\16&17&18\end{pmatrix}_{3\times3}$

(ii) $A=\begin{pmatrix}5&6\\7&8\end{pmatrix}_{2\times2}$ and $B=\begin{pmatrix}1&2\\3&4\end{pmatrix}_{2\times2}$

SOLUTION

The condition of addition is satisfied as each of the two matrices given is in the same order. Thus, placing the two matrices in a horizontal line we get,

(i) $A+B=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}_{3\times3}+\begin{pmatrix}10&11&12\\13&14&15\\16&17&18\end{pmatrix}_{3\times3}$

Adding the corresponding elements in each of the matrices we get,

$$\begin{pmatrix}1+10&2+11&3+12\\4+13&5+14&6+15\\7+16&8+17&9+18\end{pmatrix}=\begin{pmatrix}11&13&15\\17&19&21\\23&25&27\end{pmatrix}_{3\times3}$$

(ii) $A+B=\begin{pmatrix}5&6\\7&8\end{pmatrix}_{2\times2}+\begin{pmatrix}1&2\\3&4\end{pmatrix}_{2\times2}$

EXAMPLE 2. Find the sum of addition of the following three matrices :

$A=\begin{bmatrix}1&2&3&4\\5&6&7&8\end{bmatrix}_{2\times4}$, $B=\begin{bmatrix}8&9&7&6\\5&4&3&2\end{bmatrix}_{2\times4}$ and $C=\begin{bmatrix}4&3&2&1\\5&7&6&9\end{bmatrix}_{2\times4}$

SOLUTION

The condition of matrix addition is satisfied as each of the given matrices is of the same order of 2x4. Thus, placing the three matrices in a horizontal line we get,

$$A+B+C = \begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\end{bmatrix}_{2\times4} + \begin{bmatrix}8 & 9 & 7 & 6\\ 5 & 4 & 3 & 2\end{bmatrix}_{2\times4} + \begin{bmatrix}4 & 3 & 2 & 1\\ 5 & 7 & 6 & 9\end{bmatrix}_{2\times4}$$

$$= \begin{bmatrix}1+8+4 & 2+9+3 & 3+7+2 & 4+6+1\\ 5+5+5 & 6+4+7 & 7+3+6 & 8+2+9\end{bmatrix}$$

$$= \begin{bmatrix}13 & 14 & 12 & 11\\ 15 & 17 & 16 & 19\end{bmatrix}_{2\times4}$$

Properties of Matrix Addition

It may be noted that matrix addition has the following important properties of which one may take advantage in the matter of computations :

(a) It is commutative. This means, $A + B = B + A$

(b) It is associative. This means $(A + B) + C = A + (B + C)$

(c) It has additive identity. This means, $A + O = O + A = I$

(d) It has additive inverse. This means, $A + -A = -A + A = O$

(ii) Subtraction of Matrices

Condition necessary

Both the matrices i.e. the subtrahenal and the minuend matrices must be equivalent to each other. This means that each of the matrices must have equality in respect of the number of their rows and columns.

Procedure

(a) Place both the matrices in a horizontal line and put a –ve sign between the minuend and the subtrahenal matrices.

(b) Subtract the elements of the subtrahenal matrix from their corresponding elements in the minuend matrix and put their sums in the same order.

EXAMPLE 3. Subtract the matrix, B from the matrix, A where,

(i) $A = \begin{pmatrix}6 & 8\\ 2 & -3\end{pmatrix}_{2\times2}$ and $B = \begin{pmatrix}2 & 6\\ 5 & -2\end{pmatrix}_{2\times2}$ (ii) $A = \begin{pmatrix}9 & 8 & 7\\ 6 & 5 & 4\\ 1 & 2 & 3\end{pmatrix}_{3\times3}$ and $B = \begin{pmatrix}5 & 4 & 3\\ 1 & 2 & 3\\ 5 & 6 & 7\end{pmatrix}_{3\times3}$

SOLUTION

The condition of matrix subtraction is satisfied as both the matrices given, are of the same order. Now, placing both the matrices in a horizontal line as follows, we get

$$\text{(i) } A-B = \begin{pmatrix}6 & 8\\ 2 & -3\end{pmatrix}_{2\times2} - \begin{pmatrix}2 & 6\\ 5 & -2\end{pmatrix}_{2\times2}$$

$$= \begin{pmatrix}6-2 & 8-6\\ 2-5 & -3+2\end{pmatrix}_{2\times2} = \begin{pmatrix}4 & 2\\ -3 & -1\end{pmatrix}_{2\times2}$$

$$\text{(ii) } A-B = \begin{bmatrix}9 & 8 & 7\\ 6 & 5 & 4\\ 1 & 2 & 3\end{bmatrix}_{3\times3} - \begin{bmatrix}5 & 4 & 3\\ 1 & 2 & 3\\ 5 & 6 & 7\end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix}9-5 & 8-4 & 7-4\\ 6-1 & 5-2 & 4-3\\ 1-5 & 2-6 & 3-7\end{bmatrix} = \begin{bmatrix}4 & 4 & 4\\ 5 & 3 & 1\\ -4 & -4 & -4\end{bmatrix}_{3\times3}$$

EXAMPLE 4. Find the sum of the following operations on the matrices, A + B–C, where

$$A = \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}_{3\times3}, B = \begin{pmatrix}2 & 5 & 3\\ 3 & 1 & 2\\ 1 & 2 & 1\end{pmatrix}_{3\times3} \text{ and } C = \begin{pmatrix}4 & 6 & 3\\ 3 & 5 & 4\\ 3 & 0 & 1\end{pmatrix}_{3\times3}$$

SOLUTION

The conditions of both addition and subtraction of matrices are satisfied as all of the matrices given are of the same order 3x3.

Placing the matrices given, in a horizontal line we get,

$$A + B - C = \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix} + \begin{pmatrix}2 & 5 & 3\\ 3 & 1 & 2\\ 1 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 6 & 3\\ 3 & 5 & 4\\ 3 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1+2-4 & 1+5-6 & 1+3-3\\ 1+3-3 & 2+1-5 & 3+2-4\\ 1+1-3 & 4+2-0 & 9+1-1\end{pmatrix} = \begin{pmatrix}-1 & 0 & 1\\ 1 & -2 & 1\\ -1 & 6 & 9\end{pmatrix}_{3\times3}$$

(iii) Multiplication of Matrices

There can be two types of multiplication with the matrices. They are :

(a) Scalar multiplication, and

(b) Multiplication proper.

These are explained here as under :

(a) Scalar multiplication : When each element of a matrix is multiplied by a constant called a scalar, it is called scalar multiplication.

Condition necessary

No condition as to the order of a matrix is necessary except that the scalar quantity must have been given.

Procedure

(i) Write the scalar first, and then place the given matrix adjacent to it without putting any algebraic sign between them.

(ii) Multiply each element of the given matrix by the scalar given, and put the respective products in the same order.

EXAMPLE 5. Find the product of the scalar multiplication with the following :

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}_{3\times4} \quad \text{and } K = 5.$$

SOLUTION

Placing the scalar K and the matrix A in the multiplication form, we get,

$$KA = 5\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}_{3\times4}$$

Multiplying each element of the matrix by the scalar 5, we get,

$$KA = \begin{pmatrix} 5\times1 & 5\times2 & 5\times3 & 5\times4 \\ 5\times5 & 5\times6 & 5\times7 & 5\times8 \\ 5\times9 & 5\times10 & 5\times11 & 5\times12 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 10 & 15 & 20 \\ 25 & 30 & 35 & 40 \\ 45 & 50 & 55 & 60 \end{pmatrix}_{3\times4}$$

Properties of Scalar Multiplication

(i) It is distributive over addition

This implies that K (A + B) = KA + KB.

EXAMPLE 6. From the following data prove that the scalar multiplication of matrices is distributive over addition

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{pmatrix}_{2\times3}, B = \begin{pmatrix} 4 & 5 & 7 \\ 2 & 9 & 8 \end{pmatrix}_{2\times3}, \text{ and } K = 8.$$

SOLUTION

According to the property of scalar multiplication, we have

K (A + B) = KA + KB

where,

$$K(A+B) = 8\left[\begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{pmatrix} + \begin{pmatrix} 4 & 5 & 7 \\ 2 & 9 & 8 \end{pmatrix}\right]$$

$$= 8\begin{bmatrix} 1+4 & 2+5 & 3+7 \\ 5+2 & 7+9 & 8+8 \end{bmatrix} = 8\begin{bmatrix} 5 & 7 & 10 \\ 7 & 16 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 8\times5 & 8\times7 & 8\times10 \\ 8\times7 & 8\times16 & 8\times16 \end{bmatrix} = \begin{bmatrix} 40 & 56 & 80 \\ 56 & 128 & 128 \end{bmatrix}_{2\times3}$$

And

$$KA + KB = 8\begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{pmatrix} + 8\begin{pmatrix} 4 & 5 & 7 \\ 2 & 9 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 8\times1 & 8\times2 & 8\times3 \\ 8\times5 & 8\times7 & 8\times8 \end{pmatrix} + 8\begin{pmatrix} 8\times4 & 8\times5 & 8\times7 \\ 8\times2 & 8\times9 & 8\times8 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 16 & 24 \\ 40 & 56 & 64 \end{pmatrix} + \begin{pmatrix} 32 & 40 & 56 \\ 16 & 72 & 64 \end{pmatrix}$$

$$= \begin{pmatrix} 8+32 & 16+40 & 24+56 \\ 40+16 & 56+72 & 64+64 \end{pmatrix}$$

$$= \begin{pmatrix} 40 & 56 & 80 \\ 56 & 128 & 128 \end{pmatrix}_{2\times3}$$

Thus, it is proved that K (A + B) = KA + KB

(b) Multiplication proper (or MULTIPLICATION OF MATRICES)

The multiplication between two matrices is possible only when the number of columns of the Ist matrix is equal to the number of rows of the 2nd matrix. In other words a matrix A is conformable to another matrix B for multiplication i.e. AB exists, only when the number of columns in A equals to the number of rows in B.

Procedure

(i) Place the matrices in a horizontal line without putting any sign between them.

(ii) Multiply each element of the first row of the multiplicand by the corresponding element of the first column of the multiplier, and get them totalled to obtain the first element of the first row of the product matrix.

(iii) Similarly, multiply each element of the first row of the multiplicand by the corresponding element of the nth column of the multiplier and get them totalled to obtain the nth element of the first row of the product matrix.

(iv) Continue the above procedure to obtain the elements of the other rows of the product matrix.

EXAMPLE 7. Find the product of the two matrices A and B where,

$$\text{(i) } A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}_{2\times2}, B = \begin{pmatrix} 4 \\ -1 \end{pmatrix}_{2\times1} \quad \text{(ii) } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 \\ 3 & -4 \\ -5 & -6 \end{pmatrix}$$

SOLUTION

The condition of multiplication proper is satisfied by the given matrices, since the number of columns in the multiplicand matrix A (prefactor) is equal to the number of rows in the multiplier matrix B (post factor).

Placing the matrices in a horizontal line we get,

$$\text{(i)} \quad AB = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times4+3\times-1 \\ 2\times4+1\times-1 \end{pmatrix} = \begin{pmatrix} 4-3 \\ 8-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}_{2\times1}$$

(ii) $$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & -4 \\ -5 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times1+2\times3+3\times-5 & 1\times-2+2\times-4+3\times6 \\ 4\times1+5\times3+6\times-5 & 4\times-2+5\times-4+6\times6 \\ 7\times1+8\times3+9\times-5 & 7\times-2+8\times-4+9\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6-15 & -2-8+18 \\ 4+15-30 & -8-20+36 \\ 7+24-45 & -14-32+54 \end{pmatrix} = \begin{pmatrix} -8 & 8 \\ -11 & 8 \\ -14 & 8 \end{pmatrix}_{3\times2}$$

ILUSTRATION 1. Find the product of BA where,

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 2 & -4 & 5 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION

The condition of BA is satisfied by the given matrices as the number of columns in the prefactor matrix, B (n_1) is equal to the number of rows in the post factor matrix A (m_2).

Bringing the given matrices to the order of multiplication we get,

$$BA = \begin{pmatrix} 3 & 1 \\ 0 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 & 3 \\ 2 & -4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3\times-2+1\times2 & 3\times1+1\times-4 & 3\times3+1\times5 \\ 0\times-2+-1\times2 & 0\times1+-1\times4 & 0\times3+1-\times5 \\ 2\times-2+4\times2 & 2\times1+4\times-4 & 2\times3+4\times5 \end{pmatrix}$$

$$= \begin{pmatrix} -6+2 & 3-4 & 9+5 \\ 0-2 & 0+4 & 0-5 \\ -4+8 & 2-16 & 6+20 \end{pmatrix} = \begin{pmatrix} -4 & -1 & 14 \\ -2 & 4 & -5 \\ 4 & -14 & 26 \end{pmatrix}_{3\times3}$$

ILLUSTRATION 2. Find the product of the matrices A and B in that order, where

$$A = \begin{bmatrix} 0 & 5 & 3 \\ 2 & 1 & 4 \\ 1 & 6 & 2 \end{bmatrix}_{3\times3}, \text{ and } B = \begin{bmatrix} 7 & 3 & 2 \\ 1 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}_{3\times3}$$

SOLUTION

The condition of AB is satisfied by the given matrices, since $n_1 = m_2$. Placing the matrices in the order of multiplication we get,

$$AB = \begin{pmatrix} 0 & 5 & 3 \\ 2 & 1 & 4 \\ 1 & 6 & 2 \end{pmatrix}\begin{pmatrix} 7 & 3 & 2 \\ 1 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0\times7+5\times1+3\times3 & 0\times3+5\times5+3\times2 & 0\times2+5\times6+3\times1 \\ 2\times7+1\times4\times3 & 2\times3+1\times5+4\times2 & 2\times2+1\times6+4\times1 \\ 1\times7+6\times1+2\times3 & 1\times3+6\times5+2\times2 & 1\times2+6\times6+2\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 0+5+9 & 0+25+6 & 0+30+3 \\ 14+1+12 & 6+5+8 & 4+6+4 \\ 7+6+6 & 3+30+4 & 2+36+2 \end{pmatrix} = \begin{pmatrix} 14 & 31 & 33 \\ 27 & 19 & 14 \\ 19 & 37 & 40 \end{pmatrix}_{3\times3}$$

ILLUSTRATION 3. If $\begin{pmatrix} a & 4 \\ 2a+3b & 7 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 20 & 7 \end{pmatrix}$, find a and b.

SOLUTION. Since

$a = 7$ and $2a + 3b = 20$

$\therefore$ or $2\times7 + 3b = 20$ or $b = 2$

Properties of Multiplication Proper

It may be noted that the matrix multiplication has the following important properties of which one may take advantages in the course of computational works.

(a) *It is not commutative.* This means $AB \neq BA$ (always)

(b) It is associative. This means $(AB)C = A(BC)$

(c) It is distributive over addition. This means, $A(B+C)=AB+AC$

(iv) Division of Matrices

Condition necessary

The number of columns in the dividend matrix (n_1) must be equal to the number of rows in the divisor matrix (m_2).

Procedure

Proceed with the work of division just on the lines of multiplication proper explained above, except that each element in the dividend is to be multiplied by the reciprocal of the corresponding element of the divisor matrix (i.e. 1/e).

ILLUSTRATION 4. Divide the matrix A by the matrix B where,

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times3}, \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times2}$$

SOLUTION

Proceeding along the procedure of multiplication proper we g $A \div B = \begin{pmatrix} 4 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \div \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 4\times1/1+2\times1/3+3\times1/5 & 4\times1/2+2\times1/4+3\times1/6 \\ 4\times1/1+5\times1/3+6\times1/5 & 4\times1/2+5\times1/4+6\times1/6 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2/3+3/5 & 2+1/2+1/2 \\ 4+5/3+6/5 & 2+5/4+1 \end{pmatrix} = \begin{pmatrix} 79/15 & 3 \\ 103/15 & 17/4 \end{pmatrix}_{2\times2}$$

5. TRANSPOSE OF MATRICES

A matrix which is obtained by changing the rows into their respective columns or the columns into their respective rows is called a transposed matrix. To obtain such a matrix, the first row (R_1) of a given matrix is made the first column (C_1), the second row (R_2) is made the second column (C_2) and so on, or the first column (C_1) is made the first row (R_1), the second column (C_2) is made the second row (R_2) and so on. The transposed matrix is denoted by A' or A^t etc.

EXAMPLES : (i) If $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 3 \end{pmatrix}$; then $A' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 3 \end{pmatrix}$

If $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 9 \end{pmatrix}$; $B' = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

Note. Transpose of a transposed matrix reproduces the original matrix.

ORTHOGONAL MATRIX

A square matrix, which when multiplied by its transpose amounts to an identity matrix is called an orthogonal matrix. Thus, **if $A \times A' = I$, then A is an orthogonal matrix.**

SYMMETRIC MATRIX :

A square matrix A is said to be symmetric **if $A' = A$**

EXAMPLES : $\begin{pmatrix} 4 & 7 \\ 7 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ and $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ are symmetric matrices.

SKEW-SYMMETRIC MATRIX.

A square matrix is said to be Skew- Symmetric matrix, **if $A' = -A$.**

EXAMPLE : The matrix, $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$ is Skew-Symmetric since

$$A' = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix} = -A.$$

ILLUSTRATION 5. If $A = \begin{bmatrix} 3 & 2 & 1 \\ -5 & 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -5 & -2 \\ 3 & 1 & 8 \end{bmatrix}$ and verify that $(A+B)' = (A'+B')$

SOLUTION

We have, $A' = \begin{bmatrix} 3 & -5 \\ 2 & 0 \\ 1 & 6 \end{bmatrix}$ and $B' = \begin{bmatrix} -4 & 3 \\ -5 & 1 \\ -2 & 8 \end{bmatrix}$

$$\therefore \quad A'+B' = \begin{bmatrix} 3 & -5 \\ 2 & 0 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} -4 & 3 \\ -5 & 1 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & 1 \\ -1 & 2 \end{bmatrix}$$

Also $$A+B = \begin{bmatrix} 3 & 2 & 1 \\ -5 & 0 & -6 \end{bmatrix} + \begin{bmatrix} -4 & -5 & -2 \\ 3 & 1 & 8 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\therefore \quad (A+B)' = \begin{bmatrix} -1 & -2 \\ -3 & 1 \\ -1 & 1 \end{bmatrix}$$

Hence, $(A+B)' = A' + B'$ (Proved)

ILLUSTRATION 6. If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$, verify that $(AB)' = B'A'$

SOLUTION

We have,

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times0+3\times5 & 2\times(-1)\times1\times2+3\times0 \\ 4\times1+1\times0+0\times5 & 4\times(-1)+1\times2+0\times0 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$$

And $$B'A' = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+0\times1+5\times3 & 1\times4+0\times1+5\times0 \\ (-1)\times2+2\times1+0\times3 & (-1)\times4+2\times1+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$$

Hence, $(AB)' = B'A'$

ILLUSTRATION 7. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A + 14I = 0$

SOLUTION

We have

$$A^2=\begin{bmatrix}3&-5\\-4&2\end{bmatrix}\times\begin{bmatrix}3&-5\\-4&2\end{bmatrix}=\begin{bmatrix}3\times3+(-5)\times(-4)&3\times-5+(-5)\times2\\-4\times3+2\times(-4)&(-4)\times(-5)+2\times2\end{bmatrix}=\begin{bmatrix}29&-25\\-20&24\end{bmatrix}$$

$$-5A=(-5)\begin{bmatrix}3&-5\\-4&2\end{bmatrix}=\begin{bmatrix}-15&25\\20&-10\end{bmatrix}$$

$$-14I=(-14)\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}-14&0\\0&-14\end{bmatrix}$$

$\therefore$ $A^2-5A+14I=A^2+(-5)\times A+(-14I)$

$$=\begin{bmatrix}29&-25\\-20&24\end{bmatrix}+\begin{bmatrix}-15&25\\20&-10\end{bmatrix}+\begin{bmatrix}-14&0\\0&-14\end{bmatrix}$$

$$=\begin{bmatrix}29+(-15)+(-14)&-25+25+0\\-20+20+0&24+(-10)+(-14)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Hence, $A^2-5A+14I=0$

MATRIX POLYNOMIAL. Let $f(x)=a_0+a_1x+a_2x^2+\ldots+a_mx^m$ be a polynomial of degree m and let A be a square matrix of order n. Then, we define,

$$f(A)=a_0I_n+a_1A+a_2A^2+\ldots+a_mA^m$$

ILLUSTRATION 8. If $f(x)=x^2-5x+7$ and $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$, find $f(A)$.

SOLUTION

We have

$$f(A)=A^2-5A+7I_2$$

Now

$$A^2=\begin{bmatrix}3&1\\-1&2\end{bmatrix}\times\begin{bmatrix}3&1\\-1&2\end{bmatrix}=\begin{bmatrix}9-1&3+2\\-3-2&-1+4\end{bmatrix}=\begin{bmatrix}8&5\\-5&3\end{bmatrix}$$

$$-5A=(-5)\begin{bmatrix}3&1\\-1&2\end{bmatrix}=\begin{bmatrix}-5\times3&-5\times1\\-5\times-1&-5\times2\end{bmatrix}=\begin{bmatrix}-15&-5\\5&-10\end{bmatrix}$$

$$7I_2=7\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}7&0\\0&7\end{bmatrix}$$

$\therefore$

$$f(A)=A^2+5A+7I_2$$

$$=\begin{bmatrix}8&5\\-5&3\end{bmatrix}+\begin{bmatrix}-15&-5\\5&-10\end{bmatrix}+\begin{bmatrix}7&0\\0&7\end{bmatrix}$$

$$=\begin{bmatrix}8+(-15)+7&5+(-5)+0\\-5+5+0&3+(-10)+7\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Hence,

$$f(A)=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

ILLUSTRATION 9. Find the value of x such that $\begin{bmatrix}1&x&1\end{bmatrix}\begin{bmatrix}1&3&2\\2&5&1\\15&3&2\end{bmatrix}\begin{bmatrix}1\\2\\x\end{bmatrix}=0$

SOLUTION

We have,

$$\begin{bmatrix}1&x&1\end{bmatrix}\begin{bmatrix}1&3&2\\2&5&1\\15&3&2\end{bmatrix}\begin{bmatrix}1\\2\\x\end{bmatrix}=0$$

$$\Rightarrow\quad\begin{bmatrix}1+2x+15&3+5x+3&2+x+2\end{bmatrix}\begin{bmatrix}1\\2\\x\end{bmatrix}=0$$

$$\Rightarrow\quad\begin{bmatrix}16+2x&6+5x&4+x\end{bmatrix}\begin{bmatrix}1\\2\\x\end{bmatrix}=0$$

$$\Rightarrow\quad x^2+16x+28=0$$

$$\Rightarrow\quad(x+14)(x+2)=0$$

$$\Rightarrow\quad x=-14 \text{ or } x=-2$$

Hence, $x=-14$ or -2

ILLUSTRATION 10. There are two families X and Y. There are 4 men 6 women and 2 children in family X and 2 men, 2 women and 4 children in family Y. The recommended daily allowance for calories is : Man : 2400, woman : 1900, child : 1800 and for proteins is : Man : 55 gm, Woman : 45 gm and Child : 33 gm.

Represent the above information by matrices. Using matrix multiplication, calculate total requirement of calories and proteins for each of the two families.

SOLUTION

$$F=\begin{array}{c c}& \begin{array}{ccc}M&W&C\end{array}\\ \begin{array}{c}X\\Y\end{array} & \begin{bmatrix}4&6&2\\2&2&4\end{bmatrix}\end{array}$$

And the recommended daily allowance of calories and proteins for each member can be represented by 3x2 matrix

$$R=\begin{array}{c c}& \begin{array}{cc}\text{Calories}&\text{Proteins}\end{array}\\ \begin{array}{c}M\\W\\C\end{array} & \begin{bmatrix}2400&55\\1900&45\\1800&33\end{bmatrix}\end{array}$$

The total requirement of calories and proteins for each of the two families is given by the matrix multiplication.

$$FR=\begin{bmatrix}4&6&2\\2&2&4\end{bmatrix}\begin{bmatrix}2400&55\\1900&45\\1800&33\end{bmatrix}=\begin{bmatrix}4\times2400+6\times1900+2\times1800 & 4\times55+6\times45+2\times33\\2\times2400+2\times1900+4\times1800 & 2\times55+6\times45+2\times33\end{bmatrix}$$

$$=\begin{array}{c}A\\B\end{array}\begin{bmatrix}24,600&556\\15,800&332\end{bmatrix}$$

Hence, family X requires 25,600 calories and 556 gm proteins and family Y requires 15,800 calories and 332 gm proteins.

ILLUSTRATION 11. Use matrix multiplication of divide ₹ 30,000 in two parts such that the total annual interest at 9% on the first part and 11% on the second part amounts ₹ 3060.

SOLUTION

Let the two parts be ₹ x and ₹ $(30,000–x)$ respectively. Let A be the 1×2 matrix representing these two parts.

i.e.
$$A=\begin{array}{cc}\text{Part I}&\text{Part II}\\ \left[\ x\right.&\left.30,000-x\right]\end{array}$$

Let R denote the 2×1 matrix representing the annual interest rates of interest on two parts i.e.

$$R=\begin{array}{c}\text{Part I}\\\text{Part II}\end{array}\begin{bmatrix}0.09\\0.11\end{bmatrix}$$

The total annual interest on the two parts is given by the matrix multiplication AR.

$$\therefore \quad AR = [x\ (30,000-x)]\begin{bmatrix}0.09\\0.11\end{bmatrix}=3060$$

$$\Rightarrow \quad 0.09x+0.11\,(30,000–x)=3060 \Rightarrow \frac{9}{100}x+\frac{11}{100}(30,000–x)=3060$$

$$\Rightarrow \quad 9x+3,30,000–11x=3,06,000 \Rightarrow x=12,000$$

Hence, the two parts of ₹ 30,000 are ₹ 12,000 and 18,000 respectively.

EXERCISE (A)

1. Present the following data relating to marks secured by three students in the form of a matrix

Roll No.	Marks Secured		
	Business Mathematics	Statistics	Accounting
1	80	70	60
2	60	90	70
3	50	80	40

2. Present the following linear equations in the form of matrices :

(i) $2x+3y=13$
$5x+2y=16$

(ii) $5x+2y+3z=18$
$3x+2y+4z=16$
$2x+3y+3z=17$

(iii) $4x+3y+2z+7=0$
$3x+y–3z+5=0$

(iv) $x_1+x_2=2$
$_2+x_3=6$
$x_3+x_4=0$

3. Find A + B, A–B, 2A + 3B and 3A–2B from the following matrices :

$$A=\begin{pmatrix}1&2&3\\2&3&4\end{pmatrix}_{2\times3} \quad\text{and}\quad B=\begin{pmatrix}0&1&2\\3&2&6\end{pmatrix}_{2\times3}$$

4. (i) Find 3A–B and 3B–A, if

$$A=\begin{pmatrix}0&2&3\\2&1&4\end{pmatrix}_{2\times3} \quad\text{and}\quad B=\begin{pmatrix}7&6&3\\1&4&5\end{pmatrix}_{2\times3}$$

5. If $A=\begin{bmatrix}4&2&-1\\3&-7&1\end{bmatrix}_{2\times3}$ and $B=\begin{bmatrix}2&3\\-3&0\\-1&5\end{bmatrix}_{3\times2}$

Find where possible A + B, A–B, AB and BA, stating the reason where operations are not possible.

6. (i) If $Y=\begin{pmatrix}3&2\\2&4\end{pmatrix}_{2\times2}$ and $2X+Y=\begin{pmatrix}1&0\\-3&2\end{pmatrix}_{2\times2}$

Find the matrix X $\left[\textbf{Ans.} \begin{matrix}-1&-1\\-5/2&-1\end{matrix}\right]_{2\times2}$

(ii) If $2A+3B=\begin{pmatrix}2&7&12\\13&12&23\end{pmatrix}_{2\times3}$, and $A-2B=\begin{pmatrix}1&0&-1\\-4&-1&-6\end{pmatrix}_{2\times3}$

Find the matrices A and B. $\left[\textbf{Ans.}\begin{pmatrix}1&2&3\\2&3&4\end{pmatrix},\begin{pmatrix}0&1&2\\3&2&5\end{pmatrix}\right]$

7. Find the Product of the following matrices :

(i) $A=\begin{pmatrix}2&5\\1&3\end{pmatrix}$ and $B=\begin{pmatrix}1&-1\\-3&2\end{pmatrix}$ $\left[\textbf{Ans.}\begin{pmatrix}-13&8\\-8&5\end{pmatrix}\right]$

(ii) $A\begin{pmatrix}2&5\\1&3\end{pmatrix}$, and $B=\begin{pmatrix}4\\-1\end{pmatrix}$ $\left[\textbf{Ans.}\begin{pmatrix}1\\7\end{pmatrix}\right]$

(iii) $A=(2\ \ 3\ \ -1)$ and $B=\begin{pmatrix}1&0&2\\3&-2&4\\2&1&0\end{pmatrix}$ [**Ans.** (9–7 (16)]

8. If $A=\begin{pmatrix}5&3\\12&7\end{pmatrix}$, then verify that $A^2-12A-I_2=0$ where I_2 is an identity matrix of order 2.

9. Find ABC

If $A=(x\ \ y\ \ z)$, $B=\begin{pmatrix}1&3&2\\3&2&1\\2&1&3\end{pmatrix}$, and $C=\begin{pmatrix}x\\y\\z\end{pmatrix}$ [**Ans.** $x^2+2y^2+3z^2+3xy+2yz+4zx$]

10. If A. $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$, find A^2 and A^3.

[Ans. $A^2 = \begin{pmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{pmatrix}$, $A^3 = \begin{pmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{pmatrix}$]

11. If $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, verify $A^2 - 3A + 2I = 0$.

12. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, where $i^2 = -1$ Verify that $(A+B)^2 = A^2 + B^2$

13. For the matrix $P = \frac{1}{3}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, Verify that $PP' = I = P'P$, where P' is the transpose of P and I is the unit matrix of order 3.

14. Prove that the matrix A given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the relation

$A^2 - A(a+d) + (ad-bc)I = 0$, where I is a unit matrix of order two.

15. Find the values of a, b, c if the matrix A given by $A = \begin{bmatrix} a & 2 & 2 \\ 2 & 1 & b \\ 2 & c & 1 \end{bmatrix}$

Obeys the law $A.A' = I$, where A' is the transpose of A and I is the unit matrix of order 3.(Ans. $a = 1, b = -2, c = -2$)

16. If a matrix $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, Prove that $A^K = \begin{vmatrix} 1+2K & -4K \\ K & 1-2K \end{vmatrix}$, where K is any Positive integer.

17. If $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & 3 \\ -1 & 4 & 4 \end{bmatrix}$, Compute $A^2 . B^2$.

Ans. $\begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$

18. Using matrices A, B and C where, $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ Verify the rule : $(AB)C = A(BC)$

19. A, B, C and X are four matrices given by

$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 7 \\ -0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 11 \\ 5 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(i) Verify that : $AB = BA = I$ (I is a unit matrix of order 3)

(ii) If $X = BC$, find x_1, x_2 and x_3.

20. There are two families A and B. There are 2 men, 3 women and 1 child in the family A and 1 man, 1 woman and 2 children in the family B. The recommended daily allowance for calories is : Man : 2400, Woman : 1900, Child : 1800 and for proteins is : Man : 55 gms, Woman: 45 gms and Child : 33 gms.

Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.

[Ans. A : 12300, 2788 and B : 7900, 166]

21. A man buys 8 doz of mangoes, 10 doz of apples, and 4 doz of bananas. Mangoes Cost ₹ 18 per doz, apples ₹ 9 per doz and bananas ₹ 6 per doz. Represent the quantities bought by a row matrix and the price by a column matrix and hence find the total cost. [Ans. ₹ 258]

22. The following matrix gives the number of units of three products (P, Q and R) that can be processed per hour on three machines (A, B and C) Determine by using matrix algebra, how many units of each product can be produced, if the hours available on machines A, B and C are 54, 46 and 48 respectively. [Ans. P= 1812 units, Q = 2168 units and R = 2364 units]

23. A student has 4 places where he can take his break fast. The college canteen charges ₹ 8 for an egg roll, ₹ 3 for Halwa and ₹ 5 for a soft drink. The campus coffee house charges ₹ 10 for an egg roll, ₹ 2 for Halwa and ₹ 4.50 for a soft drink. A fast food place charges ₹ 8 for an egg roll, ₹ 4 for Halwa and ₹ 5 for a soft drink. A near by restaurant serves egg roll for ₹ 12, Halwa for ₹ 5 and a free soft drink for any order.

Represent the above information in a 4×3 matrix. The student wishes to buy 1 egg roll, 2 orders of Halwa and a soft drink. Using matrix algebra find the cost of break fast at each place.

[Ans. CC : ₹ 19, CH : ₹ 18.50, FFP : ₹ 21 & R : ₹ 22]

24. A firm produces three product A, B and C which it sell in two markets. Annual sales in units are given below :

Market	Units Sold		
	A	B	C
I	8000	4000	16000
II	7000	18000	9000

If the prices per unit of A, B and C are ₹ 2.50, ₹ 1.25 and ₹ 150 respectively and he costs per unit are ₹ 1.70, ₹ 1.20 and Re. 0.80 respectively, find the total profit in each market by using matrix algebra. [Ans. I - ₹ 17800, II-₹ 12800]

25. In a city there are 50 colleges and 400 schools. Each school and college has 18 peons, 5 clerks and 1 cashier. Each College in addition has 1 section officer and 1 librarian. The monthly salary of each of them is as follows :

Peon-₹ 3000, Clerk-₹ 5000, Cashier-₹ 6000, Section officer-₹ 7000 and Librarian-₹ 900

Using matrix notation, find

(i) total number of posts of each kind in school and colleges taken together, and

(ii) the total monthly salary bill of all the schools and colleges taken together.

[Ans. P–8100, C–2250, Cashier : 450, S.O.–50, L–50, ₹ 39050000]

26. A firm produces three products P_1, P_2 and P_3 requiring the mix-up of three materials M_1, M_2 and M_3. The per unit requirement of each product for each material is as follows :

$$
\begin{array}{cc}
 & \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\
A = \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} & \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix}
\end{array}
$$

Using matrix notations, find

(a) the total requirement of each material if the firm produces 100 units of each product (b) the per unit cost of production of each product if the per unit costs of materials M_1, M_2 and M_3 are ₹ 5, ₹ 10 and ₹ 5 respectively, and (c) the total cost of production, if the firm produces 200 units of each product.

[**Ans.** (a) 800, 900, 800 (b) 45, 65, 60 and (c) ₹ 34000]

27. Three firms A, B and C supplied 40, 35 and 25 truck loads of stones and 10, 5, 8 truck loads of sand respectively to a contractor. If the cost of stone and sand are ₹ 1200 and ₹ 500 per truck load respectively, find the total amount paid by the contractor to each of these firms, by using matrix method.

[**Ans.** A ₹ 53000, B ₹ 44500, C ₹ 34000]

28. A firm has two machines M_1 and M_2 costing ₹ 45000 and ₹ 30,000. Each has 5 years life with scrap value nil. Find depreciation of each machine for each year using matrix notation if (i) both are depreciated by sum of the years digit method, (ii) first is depreciated by sum of the years digit method and second by straight line method.

$$
\left[\textbf{Ans. } (i) \begin{bmatrix} 15,000 & 12,000 & 9,000 & 6,000 & 3,000 \\ 16,000 & 8,000 & 6,000 & 4,000 & 2,000 \end{bmatrix} \begin{array}{c} M_1 \\ M_2 \end{array}\right.
$$

$$
(ii) \begin{bmatrix} 15,000 & 12,000 & 9,000 & 6,000 & 3,000 \\ 6,000 & 6,000 & 6,000 & 6,000 & 6,000 \end{bmatrix} \begin{array}{c} M_1 \\ M_2 \end{array}\Bigg]
$$

5. DETERMINANT OF A MATRIX

Determinant of a matrix can be defined as a numerical value obtained from a square matrix of the coefficients of certain unknown variables enclosed by two bars by the process of diagonal expansion to tell upon a given algebraic system.

Consider the system of equations :

Let, $a_{11}x + a_{12}y = 0$

$a_{21}x + a_{22}y = 0$

By eliminating x and y, we get the expression as

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

Now, we can write the coefficients of the above equation in rows and express in the form of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ which is known as Determinant of Second Order. Thus, the determinant of 2nd order is defined as or 2 ×2 order and the Δ or D may be used as value of the determinant.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$$

Similarly, we can define the determinant of the order three or 3 ×3 order be,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a1 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Or $\quad = a\cdot\ (a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

Examples :

(i) Let the square matrix of order 2 of the coefficients of certain variables, x and y be as follows :

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}_{2\times2}$$

The determinant of such a matrix would be represented as

$$\Delta = \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 1\times4 - 3\times2 = 4 - 6 = -2$$

(ii) Let the square matrix of order 3 of the coefficient of certain variables x, y and z be as follows :

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times3}$$

The determinant of such a matrix would be represented by

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \text{ and its numerical value would be}$$

$$\Delta = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1\ (5\times9 - 6\times8) - 2\ (4\times9 - 6\times7) + 3\ (4\times8 - 5\times7)$$

$$= 1\ (45-48) - 2\ (36-42) + 3\ (32-35)$$

$$= -3 + 12 - 9 = 0$$

ILLUSTRATION 12. Find the determinant of the following matrices

(i) $A = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$, (ii) $B = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix}$

SOLUTION

(i) Given, $\quad A = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 7 & 2 \\ 3 & 6 \end{vmatrix}$

$$= 7\times6 - 2\times3 = 42 - 6 = 36$$

(ii) Given, $\quad B = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix} \Rightarrow |B| = \begin{vmatrix} 3 & 2 & 5 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{vmatrix}$

Expanding the above with the elements of first row we get.

$$\Delta = 3\begin{vmatrix} 5 & 6 \\ 8 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 1 \end{vmatrix} + 5\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 3\ (51-68) - 2\ (41-67) + 5\ (48-57)$$

$= 3\,(5-48) - 2\,(4-42) + 5\,(32-35)$

$= 3 \cdot 43 - (2 \cdot 38) + 5 \cdot 3$

$= -129 + 76 - 15 = -68$

6. COFACTORS AND MINOR OF ELEMENT

Cofactor

By the cofactor of an element of a determinant we mean the product of $(-1)^{i+j}$ and the minor of the concerned element (M_{ij}). Symbolically it is given by

$$\boxed{C_{ij} = (-1)^{i+j} \,.\, M_{ij}}$$

where C_{ij} = the cofactor of the element in the ith row and jth column of the determinant.

$(-1)^{i+j}$ = the factor determining the algebraic sign depending upon the number of rows (i) and number of columns (j) in which the element occurs in the determinant.

M_{ij} = the minor of the element in the ith row and jth column of the determinant.

Thus, $C_{11} = (-1)^{1+1} . M_{11}$; $C_{12} = (-1)^{1+2} M_{12}$, and $C_{13} = (-1)^{1+3} . M_{13}$

Minor

By the minor of an element of a determinant we mean the subsquare–determinant of the given determinant along which the particular element (a_{ij}) does not exit. It is obtained by deleting the row and the column on which the particular element (a_{ij}) lies. It is represented by M_{ij}, which denotes the minor of an element in the ith row and jth column of the determinant. Its value is obained by deducting the product of its non–leading diagonal elements from the product of its leading diagonal elements.

EXAMPLE

Let the determinant $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times3}$

The minors of its elements in the first row and first column (a_{11}), first row and second column (a_{12}) and in the first row and third column (a_{13}) will be respectively as follows :

(i) The minor of a_{11}, *i.e.*, $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22} \,.\, a_{33} - a_{32} \,.\, a_{23})$

(ii) The minor of a_{12}, *i.e.*, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21} \,.\, a_{33} - a_{31} \,.\, a_{23})$

(iii) The minor of a_{13}, *i.e.*, $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21} \,.\, a_{32} - a_{31} \,.\, a_{22})$

From the above analysis it may be noted that the minors and cofactors are mostly equal except that under certain cases they differ in sign only. We can write the expansion of a determinant in terms of minors and cofactors of the elements, i.e.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times3} = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

Similarly,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times3} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

The following examples will show how minors and cofactors of a determinant are determined and numerical value of a determinant is calculated.

EXAMPLE 13. Find the minors and cofactors of the following :

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}_{2\times2}$$

SOLUTION

Here, $a_{11} = 2$, $a_{12} = 3$, $a_{21} = 1$, and $a_{22} = 5$

(i) Minors

The minor of a_{11}, *i.e.*, $M_{11} = 5$, The minor of a_{12} *i.e.*, $M_{12} = 1$

The minor of a_{21} *i.e.*, $M_{21} = 3$, The minor of a_{22} *i.e.*, $M = 2$

(ii) Cofactors

The cofactors of an element (C_{ij}) is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

Thus, the cofactor of a_{11}, *i.e.*, $C_{11} = (-1)^{1+1} . M_{11} = 1\times5 = 5$;

The cofactor of a_{12}, *i.e.* $C_{12} = (-1)^{1+2} . M_{12} = -1\times1 = -1$;

The cofactor of a_{21}, *i.e.* $C_{21} = (-1)^{2+1} . M_{21} = -1\times3 = -3$;

And the cofactor of a_{22}, *i.e.* $C_{22} = (-1)^{2+2} . M_{22} = 1\times2 = 2$.

EXAMPLE 14. Determine the minors and the cofactors of the following determinant:

$$|B| = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}_{3\times3}$$

SOLUTION

Determination of the Minors and the Cofactors.

Here, $a_{11} = 2$, $a_{12} = 5$, $a_{13} = 3$

$a_{21} = 3$, $a_{22} = 1$, $a_{23} = 2$

$a_{31} = 1$, $a_{32} = 2$ and $a_{33} = 1$

(i) Minors

The minor of a_{11}, *i.e.*, $M_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = (1\times1 - 2\times2) = -3$

The minor of a_{12}, *i.e.*, $M_{12} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (3\times1 - 1\times2) = 1$

The minor of a_{13}, *i.e.*, $M_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = (3\times2 - 1\times1) = 5$

The minor of a_{21}, *i.e.*, $M_{21} = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = (5\times1 - 2\times3) = -1$

The minor of a_{22}, *i.e.*, $M_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = (2\times1 - 1\times3) = -1$

The minor of a_{23}, *i.e.*, $M_{23} = \begin{vmatrix}2 & 5\\1 & 2\end{vmatrix} = (2\times2-5\times1) = -1$

The minor of a_{31}, *i.e.*, $M_{31} = \begin{vmatrix}5 & 3\\1 & 2\end{vmatrix} = (5\times2-1\times3) = 7$

The minor of a_{32}, *i.e.*, $M_{32} = \begin{vmatrix}2 & 3\\3 & 2\end{vmatrix} = (2\times2-3\times3) = -5$

And The minor of a_{33}, *i.e.*, $M_{33} = \begin{vmatrix}2 & 5\\3 & 1\end{vmatrix} = (2\times1-3\times5) = -13$

(ii) Cofactors

The cofactors of an element (C_{ij}) is given by $C_{ij} = (-1)^{i+j}.M_{ij}$

Thus, the cofactor of a_{11}, *i.e.*, $C_{11} = (-1)^{1+1}.M_{11} = 1\times-3 = -3$

The cofactor of a_{12}, *i.e.*, $C_{12} = (-1)^{1+2}.M_{12} = -1\times1 = -1$

The cofactor of a_{13}, *i.e.*, $C_{13} = (-1)^{1+3}.M_{13} = 1\times5 = 5$

The cofactor of a_{21}, *i.e.*, $C_{21} = (-1)^{2+1}.M_{21} = -1\times-1 = 1$

The cofactor of a_{22}, *i.e.*, $C_{22} = (-1)^{2+2}.M_{22} = 1\times-1 = -1$

The cofactor of a_{23}, *i.e.*, $C_{23} = (-1)^{2+3}.M_{23} = -1\times-1 = 1$

The cofactor of a_{31}, *i.e.*, $C_{31} = (-1)^{3+1}.M_{31} = 1\times7 = 7$

The cofactor of a_{32}, *i.e.*, $C_{32} = (-1)^{3+2}.M_{32} = -1\times-5 = 5$

The cofactor of a_{33}, *i.e.*, $C_{33} = (-1)^{3+3}.M_{33} = 1\times-13 = -13$

Value of a determinant :

EXAMPLE 15. Find the value of the determinant given below using the Minor and cofactor expansion method :

$$|A| = \begin{vmatrix}2 & 3\\1 & 5\end{vmatrix}$$

SOLUTION

Determination of the value of |A| by the Minor expansion method

Given $|A| = \begin{vmatrix}2 & 3\\1 & 5\end{vmatrix}$

Expanding the Minors along the R1, and chaining the products we have,

$$\Delta = a_{11}M_{11} - a_{12}M_{12}.$$

Given $a_{11} = 2,\ a_{12} = 3,\ M_{11} = 5$ and $M_{12} = 1$

$$\Delta = (2\times5)-(3\times1) = 10-3 = 7$$

Determination of the value of |A| by cofactor expansion method:

$$\Delta = a_{11}.C_{11} + a_{12}.C_{12}$$

where, $a_{11} = 2$, and $a_{12} = 3$

$C_{11} = (-1)^{1+1}.M_{11} = 1(5) = 5$

$C_{12} = (-1)^{1+2}.M_{12} = -1(1) = -1$

We have, $\Delta = a_{11}C_{11} + a_{12}C_{12}$

Thus, $\Delta = (2\times5)+(3\times-1) = 10-3 = 7.$

Hence, the numerical value of the determinant $A = 7$

EXAMPLE 16. Find the value of the determinant |B| from the following using the Minor and cofactor expansion method :

$$|B| = \begin{vmatrix}2 & -5 & 3\\3 & -1 & 2\\1 & -2 & 1\end{vmatrix}$$

SOLUTION

Determination of the value of the determinant B, by the Minor expansion method

Given, $|B| = \begin{vmatrix}2 & -5 & 3\\3 & -1 & 2\\1 & -2 & 1\end{vmatrix}$

Here, $a_{11} = 2,\ a_{12} = -5,\ a_{13} = 3$

$M_{11} = \begin{vmatrix}-1 & 2\\-2 & 1\end{vmatrix},\ M_{12} = \begin{vmatrix}3 & 2\\1 & 1\end{vmatrix}$ And $M_{13} = \begin{vmatrix}3 & -1\\1 & -2\end{vmatrix}$

Expanding the Minors along the R_1 and chaining the products we have,

$$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Substituting the respective values in the above we have,

$$\Delta = 2\begin{vmatrix}-1 & 2\\-2 & 1\end{vmatrix} - (-5)\begin{vmatrix}3 & 2\\1 & 1\end{vmatrix} + 3\begin{vmatrix}3 & -1\\1 & -2\end{vmatrix}$$

$$= 2[(-1\times1)-(2\times-2)] + 5[(3\times1)-(2\times1)] + 3[(3\times-2)-(-1\times1)]$$

$$= 2\times3 + 5\times1 + 3\times-5 = 6 + 5 - 15 = -4$$

Determination of the value by the determinant by cofactor expansion method :

The cofactors of the det. B along its Row – 1 are :

$$C_{11} = (-1)^{1+1}.M_{11} = 1\begin{vmatrix}-1 & 2\\-2 & 1\end{vmatrix} = 1[(-1\times1)-(2\times-2)] = 3$$

$$C_{12} = (-1)^{1+2}.M_{12} = -1\begin{vmatrix}3 & 2\\1 & 1\end{vmatrix} = -1[(3\times1)-(2\times1)] = -1$$

$$C_{13} = (-1)^{1+3}.M_{13} = 1\begin{vmatrix}3 & -1\\1 & -2\end{vmatrix} = 1[(3\times-2)-(-1\times1)] = -5$$

The determinant B is given by

$$\Delta = a_{11}.C_{11} + a_{12}.C_{12} + a_{13}.C_{13}$$

Substituting the respective values in the above, we get,

$$\Delta = (2\times3) + -5\times(-1) + 3\times(-5)$$

$$= 6 + 5 - 15 = -4$$

Hence, the numerical value of the given 3x3 determinant B both the above ways = –4.

7. Sarrus Expansion Method

The numerical value of a determinant of the order 3 only can be determined by a sophisticated technique called, Sarrus expansion method.

The algorithm of the method is briefed here as under :

(i) Place the given determinant as it is and then repeat its firt two columns adjacent to its column 3.

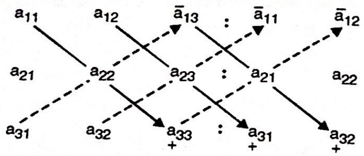
(ii) Find the products of the elements along the leading diagonal and also, those of the other two diagonals parallel to it. Get all these products summed up.

(iii) Find the products of the elements along the other 3 opposite diagonals and get them totalled.

(iv) Subtract the total of the product under the item (iii) from the total of the products under the item (ii) above, and get the difference thereof as the required value of the given determinant.

The above algorithm of the Sarrus expansion method can be clearly presented through the diagram called 'Sarrus diagram' as under :

Sarrus Diagram



The above Sarrus expansion of determinant can be expressed in the form of an equation as under :

$$\Delta = [(a_{11} \cdot a_{22} \cdot a_{33}) + (a_{12} \cdot a_{23} \cdot a_{31}) + (a_{13} \cdot a_{21} \cdot a_{32})]$$
$$- [(a_{31} \cdot a_{22} \cdot a_{13}) + (a_{32} \cdot a_{23} \cdot a_{11}) + (a_{33} \cdot a_{21} \cdot a_{12})]$$

However, the above Sarrus method is restricted to the determinants of order 3 only.

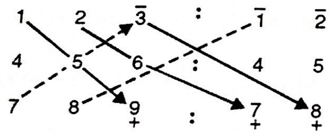
EXAMPLE 17. Using the Sarrus expansion method, evaluate the value of the determinant given below :

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

SOLUTION

Determination of the value of the |A| by the Sarrus Expansion Method.

Putting the elements of the given determinant in the Sarrus model we have :



Now, by the Sarrus equation we have,

$$\Delta = [(1\times5\times9) + (2\times6\times7) + (3\times4\times8)] - [(7\times5\times3) + (8\times6\times1) + (9\times4\times2)]$$
$$= [45 + 84 + 96] - [105 + 48 + 72]$$
$$= 225 - 225 = 0$$

Hence, the required numerical value of the determinant A = 0

EXERCISE (B)

1. Find the numerical value of the following 2×2 determinants.

(i) $\begin{vmatrix} 7 & 8 \\ 8 & 9 \end{vmatrix}$ (ii) $\begin{vmatrix} 5 & 0 \\ 8 & 2 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 7 \\ 3 & 2 \end{vmatrix}$ (iv) $\begin{vmatrix} 5 & 7 \\ 6 & 6 \end{vmatrix}$ **Ans.** [(i)–1, (ii) 10, (iii) –21, (iv) –33]

2. Find the numerical value of the following 3×3 determinants

(i) $\begin{vmatrix} 6 & 7 & 8 \\ 9 & 0 & 5 \\ 4 & 3 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 4 & 3 \\ 3 & 6 & 7 \\ 4 & 8 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 0 & 0 \end{vmatrix}$ **Ans.** [(i) 140, (ii) 0, (iii) 0, (iv) 20]

3. Find the Minors of the following determinants :

(i) $\begin{vmatrix} 8 & 9 \\ 7 & 11 \end{vmatrix}_{2\times2}$ (ii) $\begin{vmatrix} 4 & 3 & 8 \\ 6 & 7 & 5 \\ 9 & 0 & 6 \end{vmatrix}$

(iii) $\begin{vmatrix} 3 & 7 & 2 \\ 3 & 0 & 5 \\ 8 & 2 & 1 \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 5 & 7 & 2 \end{vmatrix}$

4. Find the co–factors of the following determinants :

(i) $\begin{vmatrix} 7 & 8 \\ 3 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix}$

(iii) $\begin{vmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 9 & 0 & 9 \end{vmatrix}$

5. Determine the numerical values of the following determinants using the minor expansion method :

(i) $\begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 7 & 3 \\ 7 & 6 & 0 \\ -5 & 3 & 4 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & 5 & 3 \\ 0 & 7 & 2 \\ 6 & 9 & 1 \end{vmatrix}$

Ans. [(i) 13, (ii) –19, (iii) –76, (iv) 0]

6. Find the value of the following determinants by the co–factor expansion method:

(i) $\begin{vmatrix} 3 & 5 \\ -2 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 5 & 3 \\ 0 & 2 & 5 \\ 3 & 2 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & 5 & 3 \\ 0 & 7 & 2 \\ 6 & 9 & 1 \end{vmatrix}$

Ans. [(i) 13, (ii) 49, (iii) 0, (iv) –66]

7. Find the values of the following determinants :

(i) $\begin{vmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 5 & 7 & 8 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 5 & 3 \\ 0 & 2 & -5 \\ 3 & -7 & -1 \end{vmatrix}$

8. PROPERTIES OF A DETERMINANT

Determinants have a good number of algebraic properties which help us in finding out the numerical values of the determinants at an ease and straightway. Besides, they also help us in applying the elementary operations over the row and columns of a determinant to simplify its form and arrive at the value at a quicker rate. These properties hold good for the determinants of any order but we shall probe them to the bottom only with the determinants upto the order 3.

Some such important properties are enumerated here as under :

PROPERTY-1

The value of the determinant remains unchanged even if its rows and columns are interchanged, i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof :

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by interchanging the rows and columns of A.

Then $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$

Expansion by first column

$$= a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - c_2 a_3) + c_1(a_2 b_3 - a_3 b_2)$$

$$= \Delta \text{(by Ist row expansion of } \Delta)$$

Remember : If A is a square matrix $|A'| = A$

EXAMPLE : From the following determinants verify that the value of the determinant remains unchanged even if its rows and columns are interchanged.

(i) $\begin{vmatrix} 4 & 7 \\ 9 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

SOLUTION

(i) Let $A = \begin{vmatrix} 4 & 7 \\ 9 & 3 \end{vmatrix}$, and $A' = \begin{vmatrix} 4 & 9 \\ 7 & 3 \end{vmatrix}$

Thus, $\Delta = (4\times3 - 9\times7) = -51$

$\Delta' = (4\times3 - 7\times9) = -51$

Hence $|A| = |A'|$

(ii) Let $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ and $A' = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

Then, $\Delta = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1\,(5\times9 - 8\times9) - 2(4\times9 - 7\times6) + 3(4\times8 - 7\times5) = 0$

$\Delta' = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 8 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 6 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$

$= 1(5\times9 - 6\times8) - 4(2\times9 - 6\times8) + 7\,(2\times8 - 3\times5) = 0$

Hence, $\Delta = \Delta'$ or $|A| = |A'|$ Proved

PROPERTY-2

If any two rows or columns of determinant are interchanged, the numerical value of the determinant remains the same but with the opposite sign, i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \text{ (interchanging Ist \& 3rd row)}$$

Proof.

Let $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by interchanging any two rows, say 1st and 3rd of Δ. Then

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = a_3\begin{vmatrix} b_2 & c_2 \\ a_1 & c_1 \end{vmatrix} - a_2\begin{vmatrix} b_3 & c_3 \\ a_1 & c_1 \end{vmatrix} + a_1\begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix}$$

(Expanding 1st Column)

$$= a_3(b_1 c_1 - a_1 c_2) - a_2(b_3 c_1 - a_1 c_3) + a_1(b_3 c_2 - b_2 c_3)$$
$$= -a_1(b_2 c_3 - b_3 c_2) + a_2(a_1 c_3 - b_3 c_1) - a_3(a_1 c_2 - b_1 c_1)$$
$$= -[a_1(b_2 c_3 - b_3 c_2) - a_2(a_1 c_3 - b_3 c_1) + a_3(a_1 c_2 - b_1 c_1)$$
$$= -\Delta \text{ (Expanded by Ist column)}$$

EXAMPLE 13. Prove that

$$\begin{vmatrix} 4 & 1 & 5 \\ 6 & 3 & 6 \\ 9 & 2 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 5 \\ 3 & 6 & 6 \\ 2 & 9 & 7 \end{vmatrix}$$

SOLUTION

$$\Delta = \begin{vmatrix} 4 & 1 & 5 \\ 6 & 3 & 6 \\ 9 & 2 & 7 \end{vmatrix} = 5\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} - 1\begin{vmatrix} 6 & 6 \\ 9 & 7 \end{vmatrix} + 5\begin{vmatrix} 6 & 3 \\ 9 & 2 \end{vmatrix} = -27$$

And $\Delta' = \begin{vmatrix} 1 & 4 & 5 \\ 3 & 6 & 6 \\ 2 & 9 & 7 \end{vmatrix} = \left[1\begin{vmatrix} 6 & 6 \\ 9 & 7 \end{vmatrix} - 4\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} + 5\begin{vmatrix} 3 & 6 \\ 2 & 9 \end{vmatrix}\right]$

$= [1(42-54)-4(21-12) + 5(27-12)]$

$= (27) = 27$

$\therefore \quad \Delta = -\Delta'$

PROPERTY-3

If any row or column of the determinant consists of zeroes only the value of the determinant becomes zero.

Proof.

Let $|A| = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}$

By expanding its 3rd column we have

$$\Delta = |A| = 0\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - 0\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + 0\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$= 0$

PROPERTY-4

If any two rows or columns of the determinant are identical then the value of the determinant becomes zero.

Proof.

Let $|A| = \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$

$\Rightarrow \quad |A| = a_1\begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & a_2 \\ a_3 & a_3 \end{vmatrix} + a_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1(b_2 a_3 - b_3 a_2) - b_1(a_2 a_3 - a_3 a_2) + a_1(a_2 b_3 - a_3 b_2)$

$= a_1 b_2 a_3 - a_1 b_3 a_2 + a_1 a_2 b_3 - a_1 a_3 b_2 = 0$

PROPERTY-5

If each determinant in a row or a column of a determinant is multiplied by a constant k, then the value of the new determinant k is k times the value of the original determinant, i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proof

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and Δ' be the determinant obtained by multiplying each element of a row, say second row of Δ by k. Then,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= a_1\begin{vmatrix} kb_2 & kc_2 \\ b_3 & c_3 \end{vmatrix} - ka_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ kb_2 & kc_2 \end{vmatrix}$

$= a_1(kb_2 c_3 - kb_3 c_2) - ka_2(b_1 c_3 - b_3 c_1) + a_3(kb_1 c_2 - kb_2 c_1)$

$= k[a_1(b_2 c_3 - b_3 c_2) + a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)]$

$= k\Delta$

EXAMPLE. Find out the value of the determinant $\begin{vmatrix} 5 & 4 & 5 \\ 15 & 6 & 6 \\ 10 & 9 & 7 \end{vmatrix}$ by direct method and also by using the property-5.

SOLUTION

(i) By direct method, $\Delta = \begin{vmatrix} 5 & 4 & 5 \\ 15 & 6 & 6 \\ 10 & 9 & 7 \end{vmatrix} = 5\begin{vmatrix} 6 & 6 \\ 9 & 7 \end{vmatrix} - 4\begin{vmatrix} 15 & 6 \\ 10 & 7 \end{vmatrix} + 5\begin{vmatrix} 15 & 6 \\ 10 & 9 \end{vmatrix} = 135$

(ii) Using the Prop. 5, $\Delta = \begin{vmatrix} 5 & 4 & 5 \\ 15 & 6 & 6 \\ 10 & 9 & 7 \end{vmatrix} = 5\left[1\begin{vmatrix} 6 & 6 \\ 9 & 7 \end{vmatrix} - 4\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} + 5\begin{vmatrix} 3 & 6 \\ 2 & 9 \end{vmatrix}\right]$

$= 5\times27 = 135$

PROPERTY-6

If every element in any row or column consists of the sum or difference of two quantities, then the determinant can be expressed in sum or difference of two determinants of the same order, such as

$$\begin{vmatrix} a_1+k_1 & b_1 & c_1 \\ a_2+k_2 & b_2 & c_2 \\ a_3+k_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$$

Proof

By expanding column—1 of the determinant, we have

$\begin{vmatrix} a_1+k_1 & b_1 & c_1 \\ a_2+k_2 & b_2 & c_2 \\ a_3+k_3 & b_3 & c_3 \end{vmatrix} = (a_1+k_1)(b_2 c_3 - b_3 c_2) - (a_2+k_2)(b_1 c_3 - b_3 c_1) + (a_3+k_3)(b_1 c_2 - b_2 c_1)$

$$= [a_1(b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)]$$
$$+ [k_1(b_2 c_3 - b_3 c_2 - k_2 (b_1 c_3 - b_3 c_1) + k_3(b_1 c_2 - b_2 c_1)]$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \text{(Proved)}$$

PROPERTY-7

If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.

Proof

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let $\Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and let $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then, $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \times 0$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta \text{ (Proved)}$$

PROPERTY-8

If the elements of any row or column of a determinant are multiplied in order by the cofactor C_{ij} of the corresponding elements of any other row or column, then the sum of the products thus obtained is zero.

EXANPLE. Let $\Delta = \begin{vmatrix} 4 & 7 & 3 \\ 8 & 0 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

The cofactors of the elements of the C_2 are :

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 8 & 5 \\ 2 & 1 \end{vmatrix} = -1\,(8 - 10) = 2$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 1\,(4 - 6) = -2$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 3 \\ 8 & 5 \end{vmatrix} = -1\,(20 - 24) = 4$$

...multiplying the elements of the C_3 by the cofactors of the corresponding elements of the C_2 and ...them summed up we get, $3{\times}2 + 5{\times}{-2} + 1{\times}4 = 6 - 10 + 4 = 0$.

...multiplying the elements of the C_1 by the cofactors of the corresponding elements of the C_2 ...ing them totalled we get, $4{\times}2 + 8{\times}{-2} + 2{\times}4 = 8 - 16 + 8 = 0$.

...ce, from the results of all the above examples the property of the determinant thus cited is proved.

...ICAL ILLUSTRATIONS

...LUSTRATION 18. Using the appropriate properties evaluate the determinants,

$$|A| = \begin{vmatrix} 1 & 3 & 4 \\ 5 & 15 & 16 \\ 6 & 18 & 8 \end{vmatrix}$$

SOLUTION

Given, $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 5 & 15 & 16 \\ 6 & 18 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3\times1 & 4 \\ 5 & 3\times5 & 16 \\ 6 & 3\times6 & 8 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & 4 \\ 5 & 5 & 16 \\ 6 & 6 & 18 \end{vmatrix}$

$$= 3 \times 0 = 0$$

...LUSTRATION 19. Using the relevant property, prove that,

$$\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0$$

SOLUTION

We have, $\Delta = \begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix}$

$$= \begin{vmatrix} x-z & y-x & z-y \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} R_1 \to R_1 + R_2$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} R_1 \to R_1 + R_3$$

$$= 0 \text{ (Proved)}$$

...LUSTRATION 20. Evaluate :

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix}$$

SOLUTION

We have, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a+b+c & a+b+c & a+b+c \\ b+c & a+c & a+b \end{vmatrix} R_2 \to R_2 + R_3$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ b+c & a+c & a+b \end{vmatrix}$$

$$= (a+b+c)\,(0) = \mathbf{0}$$

Hence, the value of the determinant is zero.

ILLUSTRATION 21. Using the relevant property, find the value of the following by elementary operation method :

$$|A| = \begin{vmatrix} 3 & 6 & 12 \\ 6 & 9 & 21 \\ 9 & 12 & 30 \end{vmatrix}$$

SOLUTION

Given,
$$|A| = \begin{vmatrix} 3 & 6 & 12 \\ 6 & 9 & 21 \\ 9 & 12 & 30 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 3 \\ 6 & 9 & 3 \\ 9 & 12 & 3 \end{vmatrix} C_3 \to C_3 - 3C_1$$

$$= \begin{vmatrix} 3 & 3 & 3 \\ 6 & 3 & 3 \\ 9 & 3 & 3 \end{vmatrix} C_2 \to C_2 - C_1$$

$$= 0$$

Hence, the value of $|A| = 0$.

ILLUSTRATION 22. Using the property of the determinant, prove that

$$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = (1+a+b+c)$$

SOLUTION

We have,
$$|A| = \begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$$

$$= \begin{vmatrix} 1+a+b+c & b & c \\ 1+a+b+c & 1+b & c \\ 1+a+b+c & b & 1+c \end{vmatrix} C_1 \to C_1 + C_2 + C_3$$

$$= (1+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix}$$

$$= 1+a+b+c\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & b & 1+c \end{vmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$= (1+a+b+c)\left[1\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}\right]$$

$$= (1+a+b+c) \times 1 = 1 + a + b + c$$

Hence, $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$ (Proved)

ILLUSTRATION 23. Prove that $\begin{vmatrix} a+b & c & d \\ b & a+c & d \\ b & c & a+d \end{vmatrix} = a^2(a+b+c+d)$

SOLUTION

We have,
$$|A| = \begin{vmatrix} a+b & c & d \\ b & a+c & d \\ b & c & a+d \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c+d & c & d \\ a+b+c+d & a+c & d \\ a+b+c+d & c & a+d \end{vmatrix} C_1 \to C_1 + (C_2 + C_3)$$

$$= (a+b+c+d)\begin{vmatrix} 1 & c & d \\ 1 & a+c & d \\ 1 & c & a+d \end{vmatrix}$$

$$= (a+b+c+d)\begin{vmatrix} 1 & c & d \\ 0 & a & -a \\ 0 & 0 & a \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

Expanding the above determinant along the C_1, we get,

$$\Delta = (a+b+c+d)\left[1\begin{vmatrix} a & -a \\ 0 & a \end{vmatrix}\right]$$

$$= (a+b+c+d)\,[1(a \times a - 0 \times -a)]$$

$$= (a+b+c+d)\,[1(a^2)] = \mathbf{a^2\,(a+b+c+d)}$$

Hence, $\begin{vmatrix} a+b & c & d \\ b & a+c & d \\ b & c & a+d \end{vmatrix} = a^2(a+b+c+d)$

ILLUSTRATION 24. Show that

$$\begin{vmatrix} p-q-r & 2p & 2p \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix} = (p+q+r)^3$$

SOLUTION

We have,
$$|A| = \begin{vmatrix} p-q-r & 2p & 2p \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix}$$

$$= \begin{vmatrix} p+q+r & p+q+r & p+q+r \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix} R_1 \to R_1 + R_2 + R_3$$

$$= (p+q+r)\begin{vmatrix} 1 & 1 & 1 \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix} \text{ by the property 7}$$

$$= (p+q+r)\begin{vmatrix} 1 & 0 & 0 \\ 2q & -q-r-p & 0 \\ 2r & 0 & r-p-q \end{vmatrix} \begin{matrix} C_2 \to C_2 - C_3 \\ C_3 \to C_3 - C_1 \end{matrix} \quad C_2 \to C_2 - C_3$$

Expanding the above by R_1 we get,

$$\Delta = (p+q+r)\left[1\begin{vmatrix} -q-r-p & 0 \\ 0 & -r-p-q \end{vmatrix}\right]$$

$$= (p+q+r)\,[(-q-r-p)(-r-p-q)]$$

$$= (p+q+r)(-p-q-r)^2 = (p+q+r)(p+q+r)^2 = (p+q+r)^3$$

Hence,
$$\begin{vmatrix} p-q-r & 2p & 2p \\ 2q & q-r-p & 2q \\ 2r & 2r & r-p-q \end{vmatrix} = (p+q+r)^3$$

ILLUSTRATION 25.

Prove that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

SOLUTION

Let

$$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ab \\ ca^2b^2 & abc & ac+bc \end{vmatrix} \begin{matrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to cR_3 \end{matrix}$$

$$= \frac{1}{abc}(abc)^2\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix} \begin{matrix} abc \text{ from } C_1 \\ \text{and } abc \text{ from } C_2 \end{matrix}$$

$$= abc\begin{vmatrix} bc & 1 & ab+bc+ac \\ ca & 1 & ab+bc+ac \\ ab & 1 & ab+bc+ac \end{vmatrix} \quad C_3 \to C_3 + C_1$$

$$= (abc)(ab+bc+ac)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$

$= (abc)(ab+bc+ac) \times 0 = 0$ Proved

EXERCISE (C)

1. Prove that :

(i) $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & 18 & 72 \\ 2 & 40 & 148 \\ 2 & 45 & 150 \end{vmatrix} = -12,$

(iv) $\begin{vmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 6 & 8 & 10 \end{vmatrix} = 8$

2. Prove without expansion :

(i) $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} a-b & 1 & a \\ b-c & 1 & b \\ c-a & 1 & c \end{vmatrix} = \begin{vmatrix} a & 1 & b \\ b & 1 & c \\ c & 1 & a \end{vmatrix}$

(iii) $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0,$

(iv) $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$

3. Show that : $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

4. Show that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

5. Prove that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x+y+z)(x-y)(y-z)(z-x)$

6. Prove that $\begin{vmatrix} x & y & z \\ x-y & y-z & z-x \\ y+z & z+x & x+y \end{vmatrix} = x^3+y^3+z^3-3xyz$

7. Show that $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} = 3$ where, $\omega = \sqrt[3]{1}$, and $1+\omega+\omega^2 = 0$

8. Show that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

9. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a-b)(b-c)(a-c)$

and hence deduce the linear factors of $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^4 & b^4 & c^4 \end{vmatrix}$

10. Show that $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = (xyz+1)(x-y)(y-z)(z-x)$

11. Prove that $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

12. Show that $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+z & y & z \end{vmatrix} = (x+y+z)(x-z)^2$

13. Prove that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

14. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

15. Prove that $\begin{vmatrix} b^2+c^2 & ab & ca \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

16. Prove that $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

17. $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then prove that $xyz = -1$

18. Prove that $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$

19. If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$.

Show that $abc(ab+bc+ca) = a+b+c$

20. Show that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

9. SOLUTION OF A SYSTEM OF LINEAR EQUATIONS USING DETERMINANTS (CRAMERS'S RULE)

System of equations of two unknown values

THEOREM 1. (Cramer's Rule) :

The solution of the system of equations

$$a_1 x + b_1 y = c_1 \quad ...(i)$$

$$a_2 x + b_2 y = c_2 \quad ...(ii)$$

is given by $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$,

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and $D \neq 0$.

Proof

Multiply (i) by b_2 and (ii) by b_1 and then subtracting the two equations, we get

(i) × b_2 ... $a_1 b_2 x + b_1 b_2 y = c_1 b_2$

(−) (ii) × b_1 .... $a_2 b_1 x + b_1 b_2 y = c_2 b_1$

$\Rightarrow a_1 b_2 x - a_2 b_1 x = b_2 c_1 - b_1 c_2$

$\Rightarrow x(a_1 b_2 - a_2 b_1) = (b_2 c_1 - b_1 c_2)$

$\Rightarrow \quad x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}$

Again, multiplying (i) by a_2 and (ii) by a_1 and then subtracting the two equations, we get

(i) × a_2 ... $a_1 a_2 x + a_2 b_1 y = a_2 c_1$

(−) (ii) × a_1 ....$a_1 a_2 x + a_1 b_2 y = a_1 c_2$

We get, $a_2 b_1 y - a_1 b_2 y = a_2 c_1 - a_1 c_2$

$\Rightarrow a_1 b_2 y - a_2 b_1 y = a_1 c_2 - a_2 c_1$

$\Rightarrow y(a_1 b_2 - a_2 b_1) = (a_1 c_2 - a_2 c_1)$

$\Rightarrow \quad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}$

REMEMBER :

(i) Here, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ = determinant of the coefficients of x and y

(ii) To obtain D_x, replace a_1, a_2 by c_1, c_2, respectively in D.

(iii) To obtain D_y, replace b_1, b_2 by c_1, c_2 respectively in D.

ILLUSTRATION 25. Solve the following system of equations through determinants using Cramer's Rule :

(i) $4x + 3y = 8$, $6x + 7y = 17$

(ii) $3x + 4y = 5$, $x - y = -3$

SOLUTION

(i) The given equations are :

$$4x + 3y = 8 \quad ...(i)$$

$$6x + 7y = 17 \quad ...(ii)$$

We have, $D = \begin{vmatrix} 4 & 3 \\ 6 & 7 \end{vmatrix} = 4\times7 - 6\times3 = 28 - 18 = 10$

$D_x = \begin{vmatrix} 8 & 3 \\ 17 & 7 \end{vmatrix} = 8\times7 - 17\times3 = 56 - 51 = 5$

And $D_y = \begin{vmatrix} 4 & 8 \\ 6 & 17 \end{vmatrix} = 4\times17 - 6\times8 = 68 - 48 = 20$

Applying the Cramer's rule we get,

$$x = \frac{D_x}{D} = \frac{5}{10} = \frac{1}{2} \text{ or } 0.5$$

$$y = \frac{D_y}{D} = \frac{20}{10} = 2$$

Hence, $x = 0.5$ and $y = 2$

(ii) The given equations are :

$3x + 4y = 5.$ ..(i)

$x - y = -3$ ...(ii)

We have, $D = \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = 3\times(-1) - 1\times4 = -3 - 4 = -7$

$D_x = \begin{vmatrix} 5 & 4 \\ -3 & -1 \end{vmatrix} = 5\times(-1) - (-3)\times4 = 5 + 12 = 7$

And $D_y = \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = 3\times(-3) - 1\times5 = -9 - 5 = -14$

Applying the Cramer's rule we get,

$$x = \frac{D_x}{D} = \frac{7}{-7} = -1$$

And $$y = \frac{D_y}{D} = \frac{-14}{-7} = 2$$

Hence $x = -1$ and $y = 2$

System of equations of three unknown values

THEOREM 2. (Cramer's Rule) :

The solution of the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

is given by $$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}, \text{ where } D \neq 0$$

Here, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Proof :

Let $D = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$xD = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$

$\Rightarrow$ $xD = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2x & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$ $C_1 \to C + yC_2 + zC_3$

$\Rightarrow$ $xD = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

[using eq. (i) and (ii) and (iii) as $a_1x_1+b_2y+c_1z = d_1$ and so on.]

$\Rightarrow$ $xD = D_x \therefore x = \frac{D_x}{D}$

Similarly we can find that

$yD = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

$\Rightarrow$ $yD = D_y$ and $y = \frac{D_y}{D}$

And $zD = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$\Rightarrow$ $zD = D_z$ and $z = \frac{D_z}{D}$

Hence, $$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

REMEMBER :

(i) Here, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(ii) To obtain D_x replace a_1, a_2, a_3 by d_1, d_2, d_3 respectively in D.

(iii) To obtain D_y replace b_1, b_2, b_3 by d_1, d_2, d_3 respectively in D.

(iv) To obtain D_z replace C_1, C_2, C_3 by d_1, d_2, d_3 respectively in D.

ILLUSTRATION 26. Solve the following system of equations by using Cramer's rule.

(i) $3x - 4y + 5z = -6$
$x + y - 2z = -1$
$2x + 3y + z = 5$

(ii) $2y - 3z = 0$
$x + 3y = -4$
$3x + 4y = 3$

SOLUTION

(i) We have,

$$D = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3\,(1\times1-(-2)\times3) - (-4)\,(1\times1 - (-2)\times2) + 5\,(1\times3 - 1\times2) = 3(1+6) + 4\,(1+4) + 5\,(3-2) = 46$$

$$D_x = \begin{vmatrix} -6 & -4 & -5 \\ -1 & 1 & -2 \\ 5 & 3 & 1 \end{vmatrix} = -6\,(1\times1-(-2)\times3) - (-4)\,(-1\times1 - 5\times-2) + 5\,(-1\times3-1\times5) = -6(1+6) + 4\,(-1+10) + 5\,(-3-5) = -46$$

$$D_y = \begin{vmatrix} 3 & -6 & 5 \\ 1 & -1 & -2 \\ 2 & 5 & 1 \end{vmatrix} = 3\,(-1\times1-(-2)\times5)-(-6)\,(1\times1-(-2)\times2)+5\,(1\times5-(-1)\times2) = 3(-1+10) + 6\,(1+4) + 5\,(5+2) = 92$$

$$D_z = \begin{vmatrix} 3 & -4 & -6 \\ 1 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = 3\,(1\times5-(-1)\times3) - (-4)\,(1\times5-(-1)\times2)+(-6)\,(1\times3 - 1\times2) = 3(5+3) + 4\,(5+2) - 6\,(3-2) = 46$$

Applying Cramer's rule we have,

$$x = \frac{D_x}{D} = \frac{-46}{46} = -1$$

$$y = \frac{D_y}{D} = \frac{92}{46} = 2$$

$$z = \frac{D_z}{D} = \frac{46}{46} = 1$$

Hence, $x = -1$, $y = 2$ and $z = 1$

(ii) The given system of equation can be expressed as follows :

$$0.x + 2y - 3z = 0$$
$$x + 3y + 0.z = -4$$
$$3x + 4y + 0.z = 3$$

Now, by expanding column 3 we have,

$$D = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} = (-3)\,(1\times4 - 3\times3) = -3\times-5 = 15$$

$$D_x = \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} = (-3)\,(-4\times4 - 3\times3) = -3\,(-16-9) = 75$$

$$D_y = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix} = -3\,(1\times3 - (3\times-4)) = -3\,(3+12) = -45$$

$$D_z = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix} = -2\,(1\times3 - (3\times-4)) = -2\,(3+12) = -30$$

(by expanding Row –1)

Applying Cramer's rule we have,

$$x = \frac{D_x}{D} = \frac{75}{15} = 5,$$

$$y = \frac{D_y}{D} = \frac{-45}{15} = -3$$

$$z = \frac{D_z}{D} = \frac{-30}{15} = -2$$

Hence, $x = 5$, $y = -3$ and $z = -2$

Consistent and Inconsistent Systems of Equations

A system of equation is said to be consistent or inconsistent according as its solution exists or not. **If a system has a solution, it is said to be consistent and if a system does not have a solution, it is said to be inconsistent.**

Example :

$$x + y = 5$$
$$3x + 3y = 9$$

The equation has no solution and such system of equation is said to be inconsistent equations.

Dependent equations

A system of equations is said to be dependent if it has an infinite number of solutions.

Conditions for Consistency

(a) In case of two unknown variables

We have, $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$

In this case we have the following conditions :

(i) If $D \neq 0$, then the system is consistent and has a unique solution.

(ii) If $D = 0$, $D_x = 0$ and $D_y = 0$, the system has an infinite number of solutions.

(iii) If $D = 0$ and $D_x \neq 0$ or $D_y \neq 0$ the system is inconsistent.

(b) In case of three unknown variables

We have, $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$ and $z = \dfrac{D_z}{D}$

Now the following different cases arises :

(i) If $D \neq 0$ then unique solution exists.

(ii) If D = 0 and atleast one of the determinants D_x, D_y and D_z is non–zero, the given solution is inconsistent.

(iii) If $D = D_x = D_y = D_z = 0$ then the system may or may not be consistent, as in this case the values of x,y, z assumes an **indeterminante** form.

ILLUSTRATION 27. Show that the following system of equation is inconsistent.

$$x + 3y = 2, \quad 2x + 6y = 7$$

SOLUTION

We have, $D = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 1\times6 - 2\times3 = 0$

$$D_x = \begin{vmatrix} 2 & 3 \\ 7 & 6 \end{vmatrix} = 2\times6 - 7\times3 = -9, \neq 0$$

Now, $D = 0$ and $D_x \neq 0$

Hence, the given system is inconsistent.

ILLUSTRATION 28. Show that the following system of equations is inconsistent.

$$2x - y + z = 4, \quad x + 3y + 2z = 12, \quad 3x + 2y + 3z = 10$$

SOLUTION

We have,

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = [2(9-4) + 1(3-6) + 1(2-9)] = 0$$

$$D_x = \begin{vmatrix} 4 & -1 & 1 \\ 12 & 3 & 2 \\ 10 & 2 & 3 \end{vmatrix} = [4(9-4) + 1(36-20) + 1(24-30)] = 30 \neq 0$$

Now, $D = 0$ and $D_x \neq 0$

Hence, the given system is inconsistent.

ILLUSTRATION 29. 7 pens and 30 pencils together cost ₹ 95, while 5 pens and 40 pencils together cost ₹ 105. Find the cost of a pen and that of a pencil using Cramer's rule.

SOLUTION

Let the price of a pen be ₹ x and that of a pencil be ₹ y. Arranging the date in the form of equation we have,

$$7x + 30y = 95 \quad \text{...(i)}$$
$$5x + 40y = 105 \quad \text{...(ii)}$$

We have $D = \begin{vmatrix} 7 & 30 \\ 5 & 40 \end{vmatrix} = 7\times40 - 5\times30 = 280 - 150 = 130$

$$D_x = \begin{vmatrix} 95 & 30 \\ 105 & 40 \end{vmatrix} = 95\times40 - 105\times30 = 3800 - 3150 = 650$$

And $D_y = \begin{vmatrix} 7 & 95 \\ 5 & 105 \end{vmatrix} = 7\times105 - 5\times95 = 735 - 475 = 260$

Applying the Cramer's rule we get,

$$x = \frac{D_x}{D} = \frac{650}{130} = ₹\,5$$

and $y = \frac{D_y}{D} = \frac{260}{130} = ₹\,2$

Hence, the price of the pen is ₹ 5 and that of pencil is ₹ 2.

10. APPLICATION OF CRAMMER'S RULE IN ECONOMICS

Determinants and cramer's Rule are important tools for solving many problems in business and economy. Especially for searching an optimal solution of the maximization of profit or minimization of cost problems, it can often apply.

ILLUSTRATION 30. The equilibrium conditions for two related markets i.e. price of prone and price Rohu fish are given below:

$$18x - y = 87$$
$$-2x + 36y = 98$$

If price of Rohu is x and the price of Prone is y, find the equilibrium price for each market $(\bar{y}, \bar{x})$

SOLUTION

The following systems of linear equations in matrix form.

$$\begin{bmatrix} 18 & -1 \\ -2 & 36 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 87 \\ 98 \end{bmatrix}$$

$$D = \begin{vmatrix} 18 & -1 \\ -2 & 36 \end{vmatrix} = 18 \times 36 - (-2x - 1) = 646$$

$$D_x = \begin{vmatrix} 87 & 1 \\ 98 & 36 \end{vmatrix} = 87 \times 36 - (-1 \times 98) = 3230$$

$$D_y = \begin{vmatrix} 18 & 87 \\ -2 & 98 \end{vmatrix} = 18 \times 98 - (-2 \times 87) = 1938$$

$$x = \frac{D_x}{D} = \frac{3230}{346} = 5$$

$$y = \frac{D_y}{D} = \frac{1938}{646} = 3$$

Thus, the equilibrium price for each market of $(\bar{y}, \bar{x}) = (3, 5)$.

ILLUSTRATION 31. The investment – Savings (IS) and liquidity – Money (LM) equations can be reduced to the form

$$0.4Y + 150i = 209$$
$$0.1Y - 250i = 35$$

Find the equilibrium level of income Y and rate of interest.

Note: The IS and LM equations are used together to solve for equilibrium level of output (Y) and the equilibrium interest rate (i) in the economy, known as IS – LM model.

SOLUTION

The following systems of linear equations in the matrix form:

$$\begin{bmatrix} 0.4 & 150 \\ 0.1 & -250 \end{bmatrix}\begin{bmatrix} Y \\ i \end{bmatrix} = \begin{bmatrix} 209 \\ 35 \end{bmatrix}$$

$$D = \begin{vmatrix} 0.4 & 150 \\ 0.1 & -250 \end{vmatrix} = 0.4 \times (-250) - (0.1 \times 150) = -115$$

$$Dy = \begin{vmatrix} 209 & 150 \\ 35 & -250 \end{vmatrix} = 209 \times (-250) - (35 \times 150) = -57500$$

$$Di = \begin{vmatrix} 0.4 & 209 \\ 0.1 & 35 \end{vmatrix} = 0.4 \times 35 - 0.1 \times 209 = -6.9$$

$$Y = \frac{Dy}{d} = \frac{-57500}{-115} = 500$$

$$i = \frac{Di}{D} = \frac{-6.9}{-115} = 0.06$$

The equilibrium level of income Y = 500 and rate of interest $i = 0.06$.

EXERCISE (D)

1. Solve the following system of equations by the rule given by Cramer :

(i) $2x + y = 10$, $3x + 2y = 17$

(ii) $5x + 2y = 11$, $3x + 7y = 24$

(iii) $x + 3y = 0$, $3x - y = 0$

(iv) $3x - 4y = 1$, $2x - 7y = 3$

Ans. [(i) 3, 4 (ii) 1, 3 (iii) 0, 0 (iv) $\frac{-5}{13}, \frac{-7}{13}$]

2. Using Cramer's Rule find the Solution : for the following :

(i) $2x + y + z = 7$, $3x - y - z = -2$, $x + 2y - 3z = -4$

(ii) $2x - 3y + 5z = 11$, $3x + 5y - 2z = 7$, $x + 2y - 3z = -9$

(iii) $3x + 3y - 5 = 0$, $4x - 3y = 2$

(iv) $7x - 5y = 11$, $3x + 2y = 13$, $y + 10z - 5x = 14$

(v) $2x + 3y + 5z = -9$, $x + 10y + 7z = -13$, $-5x + y + 10z = 14$

(vi) $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$

Ans. [(i) 1, 2, 3 (ii) $\frac{-3}{2}, \frac{9}{2}, \frac{11}{2}$ (iii) 1, $\frac{2}{3}$, (iv) 3, 2, 2.7, (v) −3, −1, 0 (vi) 1, 2, 3]

3. The price of 3 Accounting Books and 5 Business Mathematics books is ₹ 800 and that of 6 Accounting Books and 3 Business Mathematics books is ₹ 900. Find the Cost of an Accounting Book and a Business Mathematics book. **Ans.** [100, 100]

4. The cost of 1 kg. rice and 5 kg. potato is ₹ 105 and that of 7 kg rice and 7 kg. potato is ₹ 105. Find the cost of rice and potato per kg. **Ans.** [₹ 10, 5]

5. The perimeter of a triangle is 60 cm. The longest side exceeds the shortest one by 10 cm. and the sum of the lengths of the longest and the shortest ones is twice the length of the other side. Find the lengths of the sides. **Ans.** [25, 20, 15]

6. The sum of the three numbers is 80. If we multiply the 1st by 2 and add the 2nd number and substract the 3rd we get 92. If we multiply the 1st by 3 and add the 2nd and 3rd to it, we get 184. Find the numbers. **Ans.** [52, 8, 20]

7. An Automobile Company uses three types of steel S_1, S_2 and S_3 for producing three types of Cars C_1, C_2 and C_3. Steel requirements (in tons) for each type of Cars are given below :

		Cars	
Steel	C_1	C_2	C_3
S_1	2	3	4
S_2	1	1	2
S_3	3	2	1

Using Cramer's Rule, find the number of Cars of each type of which can be produced using 29, 13 and 16 tons of steel of three types respectively.

8. Solve the following system of equations by Cramer's rule :

$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$, $\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$ and $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$ [**Ans.** $x = 2$, $y = 3$, $z = 4$]

9. Solve the following equations by the Cramer's rule.

(i) $2y - 3z = 0$, $3x + 4y = 37$, $x + 3y = 04$

(ii) $x + 6y = 5z$, $x + z = 6y$, $5x + 6y - 4z = 24$

Ans. [(i) 5, −3, −2, (ii) 4, 6, 8]

10. The cost of 3 kg. apples, 2 kg. grapes and 5 kg. oranges is ₹ 64. The cost of 2 kg. apples, 2 kg. grapes and 3kg. oranges is ₹ 44. Again, the cost of 5 kg. apples, 3 kg. grapes and 2 kg. oranges is ₹ 54. Find the cost of apple, grape and orange per kg.

Ans. [Apples–₹ 4, Grapes–₹ 6, Oranges–₹ 8]

11. ADJOINT OF A SQUARE A MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be co–factor of a_{ij} in |A|. Then, the transpose of the matrix of co–factors of elements of A is called the adjoint of A and is denoted by Adj. A.

Thus, $\text{Adj. } A = [C_{ij}]'_{n \times n} = [C_{ji}]_{n \times n}$ or $\text{Adj. } A = (CF)^t$

When CF is the matrix of the co–factors.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then Adj. } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^t = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

where C_{ij} denotes the co–factor of a_{ij} in |A|.

EXAMPLE. Find the Adjoint of the following matrices :

(i) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{bmatrix}$

SOLUTION

(i) We have, $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$

The co–factors of the elements of |A| are given by

$$C_{11} = (+)\,3 = 3,\quad C_{12} = (-)1 = -1,\quad C_{21} = (-)5 = -5 \text{ and } C_{22} = (+)2 = 2$$

and $$CF = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } A = (CF)^t = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

(ii) We have $$|A| = \begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{vmatrix}$$

The cofactors of the elements of |A| is given by

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1,\qquad C_{12} = -\begin{vmatrix} 0 & 1 \\ -4 & 3 \end{vmatrix} = -4,\qquad C_{13} = \begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8$$

$$C_{21} = -\begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = 26,\qquad C_{22} = \begin{vmatrix} 1 & 4 \\ -4 & 3 \end{vmatrix} = 19,\qquad C_{23} = -\begin{vmatrix} 1 & 2 \\ -4 & 5 \end{vmatrix} = 3$$

$$C_{31} = \begin{vmatrix} -2 & 4 \\ 2 & 1 \end{vmatrix} = -10,\qquad C_{32} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1,\qquad C_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$$

$$\therefore \quad \text{Adj. } A = (CF)' = \begin{vmatrix} 1 & -4 & 8 \\ 26 & 19 & 3 \\ -10 & -1 & 2 \end{vmatrix}^t = \begin{vmatrix} 1 & 26 & -10 \\ -4 & 19 & -1 \\ 8 & 3 & 2 \end{vmatrix}$$

THEOREM–I. If A is a square matrix of order n, then

A. (Adj. A) = (Adj. A) A = |A|. I_n

Verification

If $$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$ verify that

$$A(\text{Adj. } A) = (\text{Adj. } A).A = |A|.\ I_3$$

SOLUTION

We have :

$$|A| = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} = [1(-28+30) - 1\times(-18-0)] = 20$$

Now the co–factors of the elements of |A| is given by

$$C_{11} = \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = 2,\quad C_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21,\quad C_{13} = \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} = -18$$

$$C_{21} = \begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = 6,\ C_{22} = \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7,\ C_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 4,\ C_{32} = \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} = -8,\ C_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4$$

$$\therefore \text{Adj. } A = (CF)^t = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}^t = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

So, $$A.(\text{Adj.}A) = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \times \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I_3$$

Thus, A(Adj. A) = |A| I_3

Further, $$(\text{Adj. } A)\ A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 20\ I_3$$

Procedure to find the Adjoint of a matrix :

(i) Convert the matrix to the form of its determinant i.e., represent A as |A|.

(ii) Find all the cofactors of the determinant matrix by $C_{ij} = (-1)^{i+j}.M_{ij}$, or by multiplying the respective minors (M_{ij}) by the algebraic signs alternatively beginning with the + sign at e_{11}, since, $(-1)^{1+1}$ makes +1.

(iii) Construct a determinant of the cofactors thus obtained.

(iv) Transpose the determinant of the cofactors and represent the same as the adjoint of the matrix (adj. A). This transposed determinant is also called as Adjugate of a determinant.

These co-factors are normally shown in a square bracket.

ILLUSTRATION 32. Determine the adjoint of the matrix

$$A = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}_{2\times 2}$$

SOLUTION

Determination of the Adjoint of the matrix A.

Given, $$A = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} \Rightarrow \begin{vmatrix} 5 & 10 \\ 15 & 20 \end{vmatrix} = 5\times 20 - 15\times 10 = -50$$

Computation of the Cofactors :

Multiplying the respective minors by + and –signs alternatively we find the cofactors of the various elements of the above determinant of A as under :

$$C_{11} = +(20) = 20,\ C_{12} = -(15) = -15$$
$$C_{21} = -(10) = -10,\ C_{22} = +(5) = 5$$

$\therefore$ The matrix of the cofactors or CF = $\begin{bmatrix} 20 & -15 \\ -10 & 5 \end{bmatrix}$

Transposing the above matrix of the cofactors we get,

$$\text{Adj. } A = (CF)^t = \begin{bmatrix} 20 & -15 \\ -10 & 5 \end{bmatrix}$$

ILLUSTRATION 33. Find the adjoint of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

SOLUTION

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow |A| = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The cofactors of the elements of |A| is given by

$$C_{11} = +\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3,\ C_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6,\ C_{13} = +\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 6, C_{22} = +\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = -12, C_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6$$

$$C_{31} = +\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3, C_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6, C_{33} = +\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$$

$$\therefore \quad \text{Adj.} A = (CF)^t = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}^t = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

ILLUSTRATION 34. Show that the matrix A is its own adjoint when

$$A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}_{3\times 3}$$

SOLUTION

Given $\quad A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{vmatrix}$

Cofactors of |A|

$$C_{11} = +\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = +(0\times 3 - 4\times 1) = -4$$

$$C_{12} = -\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(1\times 3 - 4\times 1) = 1$$

$$C_{13} = +\begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = +(1\times 4 - 4\times 0) = 4$$

$$C_{21} = -\begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -(-3\times 3 - 4\times -3) = -3$$

$$C_{22} = +\begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = +(-4\times 3 - 4\times -3) = 0$$

$$C_{23} = -\begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = +(-4\times 4 - 4\times -3) = 4$$

$$C_{31} = +\begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = +(-3\times 1 - 0\times -3) = -3$$

$$C_{32} = -\begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = -(-4\times 1 - 1\times -3) = 1$$

And $\quad C_{33} = +\begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = +(-4\times 0 - 1\times -3) = 3$

Arranging the cofactors thus obtained in the form of a matrix we get,

$$CF = \begin{pmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{pmatrix}_{3\times 3}$$

Getting the above matrix transposed we have,

$$\text{Adj. } A = (CF)^t = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}_{3\times 3}$$

Thus, from the above we see that A = Adj. A or the matrix A is its own adjoint.

12. INVERSE OF A MATRIX

SINGULAR AND NON–SINGULAR MATRICES

A square matrix A is said to be singular if |A| = 0 and it is said to be non–singular if |A| ≠ 0.

EXAMPLE :

(i) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, then $|A| = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} = 0$

Thus, matrix A is a singular matrix

(ii) If $B = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} = -2 \neq 0$

Thus, matrix B is a Non–singular matrix

INVERTIBLE MATRIX. A non zero square matrix A of order n is said to be invertible if there exists a square matrix B of order n, such that $AB = BA = I_n$.

We can say that the inverse of A is B or $A^{-1} = B$

When, $AB = BA = I_n$, we have $A^{-1} = B$ and $B^{-1} = A$

THEOREM 2

A square matrix A is invertible if and only if A is non– singular i.e. A is invertible $\Leftrightarrow |A| \neq 0$

PROOF : Let A be non–singular, then $|A| \neq 0$

$\therefore$ According to Theorem–I, (A.(Adj A) = (Adj A).A = |A|. I_n

$$\Rightarrow A\left(\frac{1}{|A|}.\text{ Adj } A\right) = \left(\frac{1}{|A|}.\text{ Adj } A\right) A = I_n \text{ (Multiplying } \frac{1}{|A|} \text{ both the sides)}$$

$$\Rightarrow A\left(\frac{1}{|A|}\text{ Adj } A\right) = I_n \text{ or } \left(\frac{1}{|A|}.\text{ Adj } A\right) A = I_n \Rightarrow A^{-1} = \frac{1}{|A|}\text{ Adj } A$$

Thus, it shows that whenever matrix A is non–singular, then matrix A is invertible.

FORMULA FOR FINDING INVERSE OF A MATRIX (A^{-1})

From the above discussion, it is clear that the inverse of a matrix can be obtained only if it is a non–singular square matrix and the formula to find out the inverse of A matrix is

$$\boxed{A^{-1} = \frac{1}{|A|}\text{ Adj.A}}$$

Different Methods to find the Inverse of a Matrix

Broadly speaking, there are three important methods of determining the inverse of a matrix.

(i) Adjoint or Cofactor Method.

(ii) Elementary (E) operation or Gauss–Jordan Method

(iii) Short–out Method

Of these three, the adjoint method is the most popular one for its simplicity in procedure. The other two methods are bit complicated and lengthy in nature.

(i) Adjoint or Cofactors Method

Under this method, the following steps are to be taken up in turn for determining the inverse of a matrix.

(a) Evaluate the determinant of the matrix. If it is zero, then stop proceeding further for that in that case, inverse does not exist for the matrix.

(b) Find the adjoint of the matrix by the procedure detailed earlier.

(c) Divide the adjoint of the matrix by its determinant and get the quotient as the inverse of the said matrix. Symbolically, the inverse of a matrix is given by $A^{-1} = \frac{1}{|A|}$.Adj. A .

ILLUSTRATION 35. Find the inverse of the following matrices.

(i) $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$, (ii) $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$.

SOLUTION

(i) Determination of A^{-1} by adjoint method.

Given, $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \Rightarrow |A| = 3 \times 2 - 4 \times 1 = 2$

Inverse exists in the matrix as $|A| \neq 0$

We get the co–factors as under :

$C_{11} = (+)\ 2 = 2 \quad C_{21} = (-)\ 4 = -4$

$C_{12} = (-)\ 1 = -1 \quad C_{22} = (+)\ 3 = 3$

Thus, $\text{Adj. } A = (CF)^t = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}^t = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$

By the model of an inverse matrix we have

$$A^{-1} = \frac{1}{|A|}.\text{ Adj. } A = \frac{1}{2}\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1/2 & 3/2 \end{pmatrix}$$

(ii) Given, $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix} \Rightarrow |A| = \begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{matrix}$

Expanding the above determinant by the first row we get,

$$|A| = 1\begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}$$

$$= 1\ (36–25)–2\ (12–5) + 3\ (5–3)$$

$$= 11–14 + 6 = 3$$

We get the co–factors as under

$C_{11} = +\begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = 11, \quad C_{21} = -\begin{vmatrix} 2 & 12 \\ 5 & 3 \end{vmatrix} = -9, \quad C_{31} = +\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1$

$C_{12} = -\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -7, \quad C_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 9, \quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -2$

$C_{13} = +\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2, \quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -3, \quad C_{33} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$$CF = \begin{vmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\text{Thus, Adj. } A = (CF)' = \begin{pmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{pmatrix}' = \begin{pmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}\cdot \text{Adj. } A = \frac{1}{3}\begin{pmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 11/3 & -3 & 1/3 \\ -7/3 & 3 & -2/3 \\ 2/3 & -1 & 1/3 \end{pmatrix}$$

(ii) E–Operation or Gauss–Jordan Method :

Under this method, certain elementary row or column operations are made over the given matrix.

However, to determine the inverse of a matrix under this method, the following steps are to be taken up in turn :

PROCEDURE :

(a) See that it is a non–singular matrix, i.e., its determinant $\Delta \neq 0$.

(b) Take a unit matrix I of the same order as that of the given matrix, and place it adjacent to the given matrix as A : I.

(c) Perform the elementary operations on the rows, or (and) columns of both the matrices A and I simultaneously till the given matrix A is converted into a unit matrix I.

Note. The elementary row (column) operations to be performed involve the following steps :

(a) Inter–change of the rows or columns.

(b) Multiplication of a row (column) by a non–zero scalar, K (i.e. a constant), and denotation of the same by KR_i meaning that the *i*th row is multiplied by K.

(c) Replacement of the *i*th row (*j*th column) by the sum of the *i*th row (*j*th column) and K times the *i*th row (*j*th column). This operation is to be denoted by $R_i + K(R_j)$. Thus, $R_2 + 5(R_3)$ implies that to the second row we are adding 5 times the third row.

(iii) Sort–out Method

Under this method the following steps are to be taken up in turn to find out the inverse of a matrix :

(a) See that the determinant of the given matrix is not zero.

(b) Take a unit matrix of the same order as that of the given matrix.

(c) Augment the given matrix by the first column of the unit matrix and reduce to zero all the elements in the first column except the one which is to be reduced to unity (1) by dividing its row by such a number as will make it a unity.

(d) Similarly, augment the transformed matrix thus, obtained by the second column of the unit matrix and reduce to zero all the elements in the first column except the one which is to be reduced to unity by dividing the elements of the row by a suitable number.

(e) Lastly, augment again, the transformed matrix thus obtained by the third column of the unit matrix and reduce to zero all the elements in the first column except the one which is to be reduced to unity by dividing the elements of the row by a suitable number.

Now, get the matrix arrived at thus, as the desired inverse of the given matrix.

ILLUSTRATION 36. Obtain the inverse of the following matrix, and prove that the product of the matrix and its inverse results in a unit matrix. Use both Adjoint and E–operation method.

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

SOLUTION

(1) Determination of the A^{-1} by the Adjoint Method :

$$\text{Given, } A = \begin{pmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \end{vmatrix}$$

Expanding the above determinant by the first row we get,

$$|A| = 2\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 2(3\times 3-1\times 2)-2(1\times 3-3\times 2)+4(1\times 1-3)\times 3)$$

$$= 2(7)-2(-3)+4(-8) = 14+6-32 = -12 \neq 0.$$

Thus, there exists the inverse in the matrix.

Proceeding further, we get the cofactors as under :

$$C_{11} = +\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = +(9-2) = 7,\quad C_{12} = \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = -3(-6) = 3,\quad C_{13} = +\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = +(1-9) = -8$$

$$C_{21} = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = -(6-4) = -2,\quad C_{22} = +\begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} + (6-12) = -6,\quad C_{23} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -(2-6) = 4$$

$$C_{31} = +\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = -(4-12) = -8,\quad C_{32} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = -(4-4) = 0, \text{ and }\quad C_{33} = +\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = +(6-2) = 4$$

$$CF = \begin{bmatrix} 7 & 3 & -8 \\ -2 & -6 & 4 \\ -8 & 0 & 4 \end{bmatrix}, \text{ and Adj. } A = (CF)^t = \begin{bmatrix} 7 & 3 & -8 \\ -2 & -6 & 4 \\ -8 & 0 & 4 \end{bmatrix}^t = \begin{pmatrix} 7 & -2 & -8 \\ 3 & -6 & 0 \\ -8 & 4 & 4 \end{pmatrix}$$

$$\therefore\quad A^{-1} = \frac{1}{|A|}\text{ Adj. } A = \frac{1}{-12}\begin{bmatrix} 7 & -2 & -8 \\ 3 & -6 & 0 \\ -8 & 4 & 4 \end{bmatrix} = -\frac{1}{12}\begin{pmatrix} 7 & -2 & -8 \\ 3 & -6 & 0 \\ -8 & 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -7/12 & 2/2 & 8/12 \\ -3/12 & 6/12 & -0/12 \\ 8/12 & -4/12 & -4/12 \end{pmatrix} = \begin{pmatrix} -7/12 & 1/6 & 2/3 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}$$

(2) Determination of the A^{-1} by the E–operation or Gauss Jordon method :

From the above it is evident that |A| 0.

Thus, proceeding under the method as under we have,

$$A^{-1} = A : I = \begin{pmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Normalising the R_1 and R_3 by dividing them by 2 and 3 respectively we get,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 1/3 & 1 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

Replacing R_2 by (R_2) and R_3 by (R_3) we get,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} -1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1/3 \end{pmatrix}$$

Dividing R_2 by 2 and multiplying R_3 by –3/2 we get,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3/2 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -1/4 & 1/2 & 0 \\ 3/4 & 0 & -1/2 \end{pmatrix}$$

Replacing R_3 by (R_3) we get,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -1/4 & 1/2 & 0 \\ 1 & -1/2 & -1/2 \end{pmatrix}$$

Dividing R_3 by 3/2 we get,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}$$

Replacing R_1 by (R_1) we get,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} 3/4 & -1/2 & 0 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}$$

Replacing R1 by $(R_1 - 2R_3)$ we get,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix} \begin{pmatrix} -7/12 & 1/6 & 2/3 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}$$

Now that the given matrix is converted into a unit matrix, the inverse of the matrix is arrived at and it is the converted I.

Thus, $$A^{-1} = \begin{pmatrix} -7/12 & 1/6 & 2/3 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}_{3\times 3}$$

Now, it must be noticed that the answers for the A^{-1} under both the methods worked out as above appear to be the same.

(b) Proof that the product of the matrix and its inverse results in a unit matrix

We have,

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} -7/12 & 1/6 & 2/3 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}$$

Thus, $$A.A^{-1} = \begin{pmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} -7/12 & 1/6 & 2/3 \\ -1/4 & 1/2 & 0 \\ 2/3 & -1/3 & -1/3 \end{pmatrix}_{3\times 3}$$

$$= \begin{pmatrix} 2\times -7/12 + 2\times -1/4 + 4\times 2/3 . 2\times 1/6 + 2\times 1/2 + 4\times -1/3 & 2\times 2/3 + 2\times 0 + 4\times -1/3 \\ 1\times -7/12 + 3\times -1/4 + 2\times 2/3 \; 1\times 1/6 + 3\times 1/2 + 2\times -1/3 & 1\times 2/3 + 3\times 0 + 2\times -1/3 \\ 3\times -7/12 + 1\times -1/4 + 3\times 2/3 \; 3\times 1/6 + 1\times 1/2 + 3\times -1/3 & 3\times 2/3 + 1\times 0 + 3\times -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Thus, $A.A^{-1}$ Proved.

ILLUSTRATION 37. Using the sort–out method find the inverse of the following matrix, and prove that $A.A^{-1} = I$.

$$A = \begin{pmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{pmatrix}_{3\times 3}$$

SOLUTION

(i) Determination of the inverse of the matrix A by the sort–out method

Augmenting the given matrix by the first column of the unit matrix of 3rd order we have,

$$A : I = \begin{pmatrix} 2 & -2 & \bullet & 1 \\ 2 & 3 & \bullet & 0 \\ 1 & -1 & \bullet & 0 \end{pmatrix}$$

Dividing R_1 by 2 we get, $\begin{pmatrix} 1 & -1 & 2 & \bullet & 1/2 \\ 2 & 3 & 2 & \bullet & 0 \\ 1 & -1 & 1 & \bullet & 0 \end{pmatrix}$

Replacing R_2 by (R_2-2R_1) we get, $\begin{pmatrix} 1 & -1 & 2 & \bullet & 1/2 \\ 0 & 5 & -2 & \bullet & -1 \\ 1 & -1 & 1 & \bullet & 0 \end{pmatrix}$

Replacing R_3 by (R_3-R_1) we get, $\begin{pmatrix} 1 & -1 & 2 & \bullet & 1/2 \\ 0 & 5 & -2 & \bullet & -1 \\ 0 & 0 & -1 & \bullet & -1/2 \end{pmatrix}$

Now that the C_1 of the matrix is completely replaced by the C_1 of the unit matrix, we discard the C_1 of the matrix and augment the last three columns by the C_2 of the unit matrix as under :

$$\begin{pmatrix} -1 & 2 & 1/2 & \bullet & 0 \\ 5 & -2 & -1 & \bullet & 1 \\ 0 & -1 & -1/2 & \bullet & 0 \end{pmatrix}$$

Dividing R_2 by 5 we get, $\begin{pmatrix} -1 & 2 & 1/2 & : & 0 \\ 1 & -2/5 & -1/5 & : & 1/5 \\ 0 & -1 & -1/2 & : & 0 \end{pmatrix}$

Replacing R_1 by (R_1+R_2) we get, $\begin{pmatrix} 0 & 8/5 & 3/10 & : & 1/5 \\ 1 & -2/5 & -1/5 & : & 1/5 \\ 0 & -1 & -1/2 & : & 0 \end{pmatrix}$

Now that the C_1 of the above matrix is completely replaced by the C_2 of the unit matrix, we discard the C_1 and augment the matrix by the third column (C_3) of the unit matrix as under :

$$\begin{pmatrix} 8/5 & 3/10 & 1/5 & : & 0 \\ -2/5 & -1/5 & 1/5 & : & 0 \\ -1 & -1/2 & 0 & : & 1 \end{pmatrix}$$

Replacing R_1 by $(R_1 + 8/5\, R_3)$ we get, $\begin{pmatrix} 0 & -1/2 & 1/5 & : & 8/5 \\ -2/5 & -1/5 & 1/5 & : & 0 \\ -1 & -1/2 & 0 & : & 1 \end{pmatrix}$

Replacing R_2 by $(R_2-2/5\, R_3)$ we get, $\begin{pmatrix} 0 & -1/2 & 1/5 & : & 8/5 \\ 0 & 0 & 1/5 & : & -2/5 \\ -1 & -1/2 & 0 & : & -1 \end{pmatrix}$

Replacing R_3 by $-R_3$ we get, $\begin{pmatrix} 0 & -1/2 & 1/5 & : & 8/5 \\ 0 & 0 & 1/5 & : & -2/5 \\ 1 & -1/2 & 0 & : & -1 \end{pmatrix}$

Now that the C_1 of the above matrix is completely replaced by the C_3 of the unit matrix we get the desired

$$A^{-1} = \begin{pmatrix} -1/2 & 1/5 & 8/5 \\ 0 & 1/5 & -2/5 \\ 1/2 & 0 & -1 \end{pmatrix}_{3\times 3}$$

REMEMBER

(i) **If A and B are invertible square matrices of the same order, then its multiple AB is also invertible and $(AB)^{-1} = B^{-1}.A^{-1}$**

(ii) **If A is an invertible square matrix, then its transpose A' is also invertible and $(A')^{-1} = (A^{-1})'$**

(iii) **If A and B are invertible square matrices of the same order, then (Adj. AB) = (Adj. B) (Adj. A)**

(iv) **If A is an invertible square matrix, then $(Adj.A)' = (Adj.A')$**

ILLUSTRATION 38. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify the $(AB)^{-1} = B^{-1}A^{-1}$

SOLUTION

We have

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15-14 = 1 \neq 0$$

Cofactors of the elements of $|A|$ are :

$C_{11} = 5,\quad C_{12} = -7,\quad C_{21} = -2,\quad C_{22} = 3$

$\therefore$ Adj. $A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{|A|}.\text{Adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ $\qquad (\because |A| = 1)$

Further, $|B| = 54-56 = -2 \neq 0$

Cofactors of the elements of $|B|$ are :

$C_{11} = 9,\quad C_{12} = -8,\quad C_{21} = -7,\quad C_{22} = 6$

$\therefore$ Adj $B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}' = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

Hence, $B^{-1} = \dfrac{1}{|B|}$ Adj $B = \dfrac{1}{-2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

Again, $|AB| = |A|.|B| = 1\times(-2) = -2 \neq 0$

Adj. AB = (Adj. B) (Adj. A)

$$\text{Adj. } AB = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} \times \text{Adj.}AB = \frac{1}{-2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

$$\text{Also, } B^{-1}A^{-1} = B^{-1}A^{-1}\left(-\frac{1}{2}\right)\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \times \left(\frac{1}{1}\right)\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

Hence, $(AB)^{-1} = B^{-1}.A^{-1}$

ILLUSTRATION 39. Find a matrix X such that, $X.\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

SOLUTION

Let $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$

So, $|A| = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = (-3-2) = -5 \neq 0$

Thus, A is non–singular and therefore, invertible

The given matrix equation is $XA = B$

Now, $XA = B$

$\Rightarrow X(AA^{-1}) = BA^{-1}$

$\Rightarrow X.I_2 = BA^{-1}$

$\Rightarrow X = BA^{-1}$

The cofactors of elements of |A| are :

$C_{11} = -1, \quad C_{12} = -1, \quad C_{21} = -2, \quad C_{22} = 3,$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}.\text{Adj } A = -\frac{1}{5}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$

Since, $X = BA^{-1}$

$$X = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \times \left(\frac{-1}{5}\right).\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \left(\frac{-1}{5}\right)\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ -5 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \textbf{Ans.}$$

EXERCISE (E)

1. Find the transpose of each of the following matrices :

(i) $A = \begin{pmatrix} 2 & 5 \\ 6 & 15 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{pmatrix}$

(iii) $C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$ (iv) $D = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

2. Find the adjoint of each of the following matrices :

(i) $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 11 & 2 & 10 \end{pmatrix}$ (iv) $\begin{pmatrix} 4 & -2 & -1 \\ 1 & 10 & -7 \\ 2 & -4 & 1 \end{pmatrix}$

[**Ans.** (i) $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$, (ii) $\begin{pmatrix} -3 & -6 \\ 2 & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 30 & 12 & -3 \\ -20 & -1 & 2 \\ -29 & -13 & 5 \end{pmatrix}$ (iv) $\begin{pmatrix} -18 & 6 & 24 \\ -15 & 6 & 27 \\ -24 & 12 & 42 \end{pmatrix}$]

3. Find A Adj. A for the matrix

$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{pmatrix}$ [**Ans.** $\begin{pmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{pmatrix}$]

4. For the square matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$

Prove that A. Adj. A = $|A|.I_3$.

5. Find the inverse of each of the following matrices :

(i) $A = \begin{pmatrix} 3 & 8 \\ 2 & 1 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$ (iii) $C = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

[**Ans.** (i) $\frac{-1}{13}\begin{pmatrix} 1 & -8 \\ -2 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & \frac{1}{4} \\ 1 & -\frac{3}{4} \end{pmatrix}$ (iii) $\frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$]

6. Find the inverse of each of the following matrices (if it exists)

(i) $\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{pmatrix}$

[**Ans.** (i) $\begin{pmatrix} -1 & -3 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ (ii) $\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{pmatrix}$ (iii) Inverse does not exist]

7. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and show that $AA^{-1} = A^{-1}A = I$

[**Ans.** $-\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$]

8. If $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, find A^2 and A^{-1}, and show that $A^2 = A^{-1}$. [**Ans.** $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$]

9. Let the matrix A be given by $A = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$. Verify that $A^2 + 3A + 4 = 0$. Also obtain the inverse of A. [**Ans.** $-\frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$]

13. SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY THE MATRIX ALGEBRA

Simultaneous linear equations (involving variables upto the power 1 only) viz. : (x^1, y^1, z^1) consisting of upto 3 variables can be easily solved by the technique of matrix algebra using the inverse of a matrix. This technique was developed by the famous British Mathematician **Prof. Arthur Cayley** in 1857. According to this technique the following steps need to be taken up in turn for solving simultaneous equations involving upto 3 variables.

Steps.

(i) Present the given equations in three different matrices, viz. (a) matrix of the coefficients of the variables denoted by A, (b) matrix of the variables viz . x, y, z denoted by X, and (c) matrix of the constants denoted by B in the same order in which they stand in the equations as thus, AX = B.

(ii) Multiply A^{-1} on both the sides of the above matrix equation and reduce the same to the following solution equation : $X = A^{-1}B$ [$\because A^{-1}.AX = A^{-1}.B$]

(iii) Find the A^{-1} in accordance with its procedure detailed earlier.

(iv) Multiply the matrix A^{-1} by the matrix B and present the matrix of the product A^{-1}. B thus obtained vis-a-vis the matrix of the variables X. These matrices must be equivalent to each other for which their corresponding elements would be equal to each other.

(v) Locate the value of each variable in the matrix X with reference to the corresponding element in the product matrix. The following illustrations will show how simultaneous equations are solved using the technique of matrix algebra.

ILLUSTRATION 40. Using the technique of matrix algebra solve the following simultaneous equations :

$$4x + 3y = 8$$
$$6x + 7y = 17$$

SOLUTION

Presenting the given equations orderly in three different matrices as under we have,

$$\begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}, \text{ i.e. } A.X = B$$

where, $A = \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}; X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$

we have, $X = A^{-1}B$

where, $A^{-1} = \frac{1}{|A|}$. Adj. A and $|A| = \begin{vmatrix} 4 & 3 \\ 6 & 7 \end{vmatrix} = (4\times7 - 6\times3) = 10 \neq 0$

The above non-zero result of the determinant A shows that there exists an inverse in the matrix. Thus, proceeding further,

and computing the cofactors of A we get,

$$C_{11} = 7, C_{12} = -6, C_{21} = -3 \text{ and } C_{22} = 4$$

Now, Adj. A = $(CF)^t = \begin{pmatrix} 7 & -6 \\ -3 & 4 \end{pmatrix}^t = \begin{pmatrix} 7 & -3 \\ -6 & 4 \end{pmatrix}$

Thus, $A^{-1} = \frac{1}{|A|}$ Adj. $A = \frac{1}{10}\begin{pmatrix} 7 & -3 \\ -6 & 4 \end{pmatrix} = \begin{pmatrix} 0.7 & -0.3 \\ -0.6 & 0.4 \end{pmatrix}$

we have, for a solution, $X = A^{-1}B = \begin{pmatrix} 0.7 & -0.3 \\ -0.6 & 0.4 \end{pmatrix}\begin{pmatrix} 8 \\ 17 \end{pmatrix}$

$$= \begin{pmatrix} 0.7\times8 + (-0.3)\times17 \\ -0.6\times8 + 0.4\times17 \end{pmatrix} = \begin{pmatrix} 5.60 + (-5.10) \\ -4.80 + 6.80 \end{pmatrix} = \begin{pmatrix} 0.50 \\ 2.00 \end{pmatrix}$$

Thus, we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.50 \\ 2.00 \end{pmatrix} \Rightarrow \begin{matrix} x = 0.50 \\ y = 2.00 \end{matrix}$

ILLUSTRATION 41. Solve the following simultaneous equations by the matrix method :

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

SOLUTION

Representing the given equations orderly in the three relevant matrices as under we get,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$\Rightarrow$ A.X = B

where, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$

The solution of the equation is given by $X = A^{-1}B$

where, $A^{-1} = \frac{1}{|A|}$ Adj. A;

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$= 1\,(18–12)–1\,(9–3) + 1\,(4–2) = 6–6 + 2 = 2$ *i.e.* $|A| \neq 0$.

And Adj. A = $(CF)'$ *i.e.* the transpose of the matrix of cofactors of A. Now, the cofactors of A are :

$C_{11} = +\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = +\,(18–12) = 6$ $\quad C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -\,(9–3) = –6$ $\quad C_{13} = +\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = +\,(4–2) = 2$

$C\ = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -\,(9–4) = –5$ $\quad C_{22} = +\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = +\,(9–1) = 8$ $\quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = –(4–1) = –3$

$1| = +\,(23–2) = 1$ $\quad C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = –(3–1) = –2$ $\quad C_{33} = +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = +2(2–1) = 1$

$$A \quad \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}' = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\text{Thus, } A^{-1} = \frac{1}{|A|}\text{Adj.}A = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2.5 & 0.5 \\ -3 & 4 & -1 \\ 1 & -1.5 & 0.5 \end{pmatrix}$$

We have for solution, $X = A^{-1}.B$

Substituting the matrices of X, A^{-1}, and B in the above equation we get,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -2.5 & 0.5 \\ -3 & 4 & -1 \\ 1 & -1.5 & .5 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\times3-2.5\times4+0.5\times6 \\ -3\times3+4\times4-1\times6 \\ 1\times3=1.5\times4+0.5\times6 \end{pmatrix} = \begin{pmatrix} 9-10+3 \\ -9+16-6 \\ 3-6+3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Hence, by rule of equivalent matrices, we get,

$x = 2,\ y = 1$ and $z = 0$.

ILLUSTRATION 42. Solve the following linear equations by the method of matrix :

$3x + 11y = 7$

$6x + 22y = 5$

SOLUTION

Presenting the given equations in the matrix form we have,

$$\begin{pmatrix} 3 & 11 \\ 6 & 22 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \quad i.e.\ A.X = B$$

The solution equation is given by $X = A^{-1}B$

where, $X = \begin{pmatrix} x \\ y \end{pmatrix},\ A = \begin{pmatrix} 3 & 11 \\ 6 & 22 \end{pmatrix}, B = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and

$$A^{-1} = \frac{1}{|A|}\text{Adj.A}$$

Now $|A| = \begin{vmatrix} 3 & 11 \\ 6 & 22 \end{vmatrix} = (66–66) = 0$ *i.e.* $|A| = 0$

From the zero value of the |A|, it is clear that A is a singular matrix and there does not exist any inverse in the matrix.

In such a case, there are two possibilities as follows :

(i) Either the system of equations given is **dependent** which may call for infinite solutions (if Adj. $A \times B = 0$).

OR

(ii) The system of equation given is **inconsistent** which can have no solution (if Adj. $A \times B \neq 0$)

Adj. A = Transpose of the matrix of cofactors of the matrix A.

The various cofactors of A are :

$C_{11} = 22,\ C_{12} = –6,\ C_{21} = –1\ 1$ and $C_{22} = 3$

$$\text{Adj. } A = (CF)^t = \begin{pmatrix} 22 & -6 \\ -11 & 3 \end{pmatrix}^t = \begin{pmatrix} 22 & -11 \\ -6 & 3 \end{pmatrix}$$

$$\text{Thus, (Adj. A)} \times B = \begin{pmatrix} 22 & -11 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 154 & -55 \\ -42 & +15 \end{pmatrix} = \begin{pmatrix} 99 \\ -27 \end{pmatrix}$$, which is a non zero matrix i.e. ($\neq 0$)

Therefore, the given system of equation is inconsistent, and thus there is no solution for the problem given.

ILLUSTRATION 43. Determine by the matrix method the values of x and y from the following linear equations :

$x + 5y = 7$

$3x + 15y = 21$

SOLUTION

Presenting the given equations in the matrix form we get,

$$\begin{pmatrix}1 & 5\\3 & 15\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\21\end{pmatrix}\text{ i.e. A.X = B}$$

The solution equation is given by,

$$X = A^{-1}B$$

where, $A^{-1}=\frac{1}{|A|}\text{Adj.A},\quad |A|=\begin{vmatrix}1 & 5\\3 & 15\end{vmatrix}=(15-15)=0$

∴ The matrix A is a singular matrix.

In such a case, there are two possibilities as thus,

(i) Either the system of equations is dependent which may call for infinite number of solutions (if Adj. A.B = 0)

OR

(ii) The system of equations is inconsistent which can have no solution if (Adj.A) × B ≠ 0.

Now, Adj. A = Transpose of the matrix of cofactors of A.

Where, the various cofactors of A are :

$$C_{11} = 15,\ C_{12} = -3,\ C_{21} = -5, \text{ and } C_{22} = 1$$

Thus, $\text{Adj. A} = (CF)^t = \begin{pmatrix}15 & -3\\-5 & 1\end{pmatrix}^t=\begin{pmatrix}15 & -5\\-3 & 1\end{pmatrix}$

$$\therefore (\text{Adj. A})\,B=\begin{pmatrix}15 & -5\\-3 & 1\end{pmatrix}\begin{pmatrix}7\\21\end{pmatrix}=\begin{pmatrix}105 & -105\\-2 & +21\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\begin{bmatrix}\text{which is a zero}\\\text{matrix and = 0}\end{bmatrix}$$

Therefore, the system of equations given is dependent and hence, there can be infinite number of solutions for the values of x and y. Some such are :

(i) $x = 7$, $sy = 0$, (ii) $x = 2$, $y = 1$, (iii) $x = 3$, $y = 0.8$

14. Application of Matrix Inverse Method in Economic Models.

In economics, matrix algebra is used dominantly in various purposes for determining the equilibrium values of a system of linear economic equations. Some economic models are discussed here as under:

Solution of Linear Market Model.

In linear market model, Quantity demanded = Q_d is a function of price (p) of the product and quantity supply (Q_s), is an increasing function of price (p) where market clearing condition prevail. Thus, the linear market model is,

1. $Q_d = a - bp$, when $a, b > 0$
2. $Q_s = -c + dp$ when $c, d > 0$
3. $Q_d = Q_s$ (Market clearing condition)

Here, a, b, c and d are constants.

The above given linear market models can be rearranged as follows.

1. $Q_d + bp = a$ i.e. $Q_d + 0\times Q_s + bp = a$
2. $Q_2 - dp = -e$ i.e. $0\times Q_d + Q_s - dp = -c$
3. $Q_d - Q_3 = 0$ i.e. $Q_d - Q_s + 0\times p = 0$

Again, the equations can be re-arranged in matrix form as

$$\begin{bmatrix}1 & 0 & b\\0 & 1 & -d\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}Q_d\\Q_s\\p\end{bmatrix}=\begin{bmatrix}a\\-e\\0\end{bmatrix}$$

or A X = B and $X = A^{-1}.B$

Here, $|A|=\begin{vmatrix}1 & 0 & b\\0 & 1 & -d\\1 & -1 & 0\end{vmatrix}=-(b+d)$ (i.e. ≠ 0)

Thus, A^{-1} exists as $|A|\neq 0$

$$CF=\begin{bmatrix}-d & -d & -1\\-b & -b & 1\\-b & d & 1\end{bmatrix}$$

And $\text{Adj. A} = (CF)^t=\begin{bmatrix}-d & -b & -b\\-d & -b & -d\\-1 & 1 & 1\end{bmatrix}$

Now, $A^{-1}=\frac{1}{|A|}.\text{Adj.A}=\frac{1}{-(b+d)}\begin{bmatrix}-d & -b & -b\\-d & -b & d\\-1 & 1 & 1\end{bmatrix}$

2. Since, $X = A^{-1}\times B$

Thus, $\begin{bmatrix}Q_d\\Q_3\\p\end{bmatrix}=\frac{1}{-(b+d)}\begin{bmatrix}-d & -b & -b\\-d & -b & d\\-1 & 1 & 1\end{bmatrix}\times\begin{bmatrix}a\\-c\\0\end{bmatrix}$

Hence, $\begin{bmatrix}Q_d\\Q_3\\p\end{bmatrix}=\frac{1}{-(b+d)}\begin{bmatrix}-ad+bc-0\\-ad+bc-0\\-a-c+0\end{bmatrix}=\begin{bmatrix}(ad-bc)/b+d\\(ad-bc)/b+d\\(a+e)/b+d\end{bmatrix}$

Thus Equilibrium Solution is

$$\bar{Q}_d=\frac{ad-bc}{b+d},\ \bar{Q}_s=\frac{ad-bc}{b+d}\text{ and }\bar{p}=\frac{a+c}{b+d}$$

Solution to Simple National Income Model

In economics, simple National income model is given by

(1) $Y = C + I_o + G_o$

(2) $C = a + b(Y - T)$, where, $a > 0, 0 < b < 1$

(3) $T = tY$, where, $0 < t < 1$

Here, Y = National Income, C = consumption expenditure

T = Tax, t = rate of Tax, I_o = Govt. Investments and G_o = Govt. expenditure.

The above equation can be arranged as

(1) $Y - e = I_o + G_o$ i.e. $Y - C + (0 \times T) = (I_o + G_o)$

(2) $-bY + C + bT = a$ i.e. $-bY + C + (0 \times T) = a$

(3) $-tY + T = 0$ i.e. $-tY + (0 \times c) + T = 0$

In matrix form the equations are arranged as.

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & 0 \\ -t & 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I_o + G_o \\ a \\ 0 \end{bmatrix}$$

$$AX = B$$

$$|A| = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & 0 \\ -t & 0 & 1 \end{vmatrix} = 1 - b + bt \qquad (\because 0 < b < 1 \, and \, 0 < t < 1)$$

Since, $|A| \neq 0$ and thus A^{-1} exists.

Here, $B = \begin{bmatrix} I_o + G_o \\ a \\ 0 \end{bmatrix}$, and $CF = \begin{bmatrix} 1 & 1 & -b \\ b - bt & 1 & -b \\ t & t & 1-b \end{bmatrix}$

Now, $\text{Adj. } A = [CF]^t = \begin{bmatrix} 1 & b - bt & t \\ 1 & 1 & t \\ -b & -b & 1-b \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}.\text{Adj.} A = \frac{1}{1 - b + bt}\begin{bmatrix} 1 & b - bt & t \\ 1 & 1 & t \\ -b & -b & 1-b \end{bmatrix}$$

$$X = A^{-1}.B = \frac{1}{1 - b + bt}\begin{bmatrix} 1 & b - bt & t \\ 1 & 1 & t \\ -b & -b & 1-b \end{bmatrix}.\begin{bmatrix} I_0 + G \\ a \\ 0 \end{bmatrix}$$

Thus, $$\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \frac{1}{1 - b + bt}\begin{bmatrix} (I_0 + G_0) + a - 0 \\ (b - bt)(I_0 + G_0) + a - 0 \\ t(I_0 + G_0) + t.a + 0 \end{bmatrix}$$

The solution of the simple National Income Modes is

$$Y = \frac{I_0 + G_0 + a}{1 - b + bt}, C = \frac{(I_0 + G_0)(b - bt) + a}{1 - b + bt}, T = \frac{(I_0 + G_0 + a)t}{1 - b + bt}$$

15. RANK OF A MATRIX

Rank of a matrix is the order of the highest order non-singular square sub-matrix.

A number r is said to be the rank of an $m \times n$ matrix A if

(a) Every square sub-matrix of order $(r + 1)$ or more is singular, and

(b) There exists atleast one square sub-matrix of r which is non-singular.

If A is a given matrix of rank 2, then every square sub-matrix of order 3 or more is singular and there exist atleast one square sub-matrix of order 2 which is non-singular.

Note. The rank of every non-null matrix is greater than or equal to 1.

Theorem–1. The rank of a matrix is same as the rank of its transpose, *i.e.* $r(A) = r(A')$

Theorem–2. Elementary transformation do not alter the rank of matrix.

EXAMPLE 1. A is a 2×3 matrix given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}, \text{ find the rank.}$$

SOLUTION. Here, the various possible sub-matrix of A are

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ and matrix A}$$

Here, we can observe that determinant of all these Minor matrices are zero but matrix A is non-zero. Hence, the rank of matrix A is 1.

EXAMPLE 2.

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

SOLUTION

We have,

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1\,(1-4) - 2\,(2-4) + 2\,(4-2) = -3 + 4 + 4 = 5 \text{ i.e. } \neq 0$$

Thus, A is a Non-Singular matrix of order 3. Thus, the rank of the matrix A is 3.

EXAMPLE 3.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

SOLUTION

We have, $$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= 1\begin{vmatrix}4 & 5\\5 & 6\end{vmatrix} - 2\begin{vmatrix}3 & 5\\4 & 6\end{vmatrix} + 3\begin{vmatrix}3 & 4\\4 & 5\end{vmatrix}$$

$$= (24-25) - 2(18-20) + 3(15-16) = 0$$

The rank of A is less than 3. But there is atleast one minor of order 2 viz: $\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix}$, where determinant value is not equal to zero.

Hence, the rank of the matrix A is 2.

EXAMPLE 4. Find the rank of the matrix, $A = \begin{pmatrix}1 & 2 & 3\\3 & 6 & 9\\1 & 2 & 3\end{pmatrix}$

SOLUTION. The determinant of the matrix A is zero.

It is observed that every square sub-matrix of order 2 is singular. But, A is a non-null matrix. Therefore, $r(A) = 1$ or rank of the matrix A is 1.

EXERCISE (F)

1. Solve the following system of equations using the matrix method :

(i) $2x - 3y = 1$
$x + 5y = 7$

(ii) $3x + 2y = 6$
$5x + 4y = 8$

(iii) $4x - 3y = 5$
$3x - 5y = 1$

(iv) $2x - y = 13$
$-x + 2y = -11$

(v) $2x - y + 2 = 0$
$3x + 4y - 3 = 0$

[**Ans.** (i) $x = 2, y = 1$, (ii) $x = 4, y = -3$, (iii) $x = 2, y = 1$, (iv) $x = 5, y = 3$ (v) $x = -5/11, y = 12/11$]

2. Solve by the matrix method :

(i) $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

(ii) $x + y + z = 2$
$2x + 2y + 3z = 7$
$5y - z + 13 = 0$

(iii) $8x + 4y + 3z = 13$
$2x + y + z = 5$
$x + 2y + z = 5$

(iv) $2x + 2y + z = 13$
$4y + z = 17$
$-3x + 2y = 3$

(v) $x + y + z = 3$
$2x - y + z = 2$
$x - 2y + 3z = 2$

(vi) $3x + 2y + 4z = 19$
$2x - y + z = 3$
$6x + 7y - z = 17$

[**Ans.** (i) 2, –1, 1, (ii) 1, 2, 0 (iii) $x = y = -\frac{2}{3}$, $z = 7$, (iv) 1, 3, 5, (v) $x = y = z = 1$, (vi) 1, 2, 3]

3. Solve the following equations by using matrices :

$4x + 3y + 2z + 7 = 0$
$2x + y - 4z + 1 = 0$
$x - 7y - 2 = 0$

[**Ans.** $x = \frac{-182}{154}, y = z = \frac{-70}{154}$]

4. Solve the following equations by the matrix method :

(i) $4x + 3y + z = 8$
$2x + y + 4z = -4$
$3x + z = 1$

(ii) $x + y + 2z = 4$
$2x - y + 3z = 9$
$3x - y - z = 2$

[**Ans.** (i) $x = 1, y = 2, z = -2$, (ii) $x = 1, y = -1, z = 2$]

5. An amount of ₹ 5000 is put into three investments at the rates of interest of 6%, 7% and 8% per annum respectively. The total annual income is ₹ 358. If the combined income from the first two investments is ₹ 70 more than the income from the third, find the amount of each investment by using matrix algebra. [**Ans.** ₹ 1000, ₹ 2200, ₹ 1800]

6. Mr. X has invested a part of his investment in 10% bond A and a part in 15% bond B. His interest income during first year is ₹ 4000. If he invests 20% more in 10% bond A and 10% more in 15% bond B, his income during second year increases by ₹ 500. Find his initial investment and the new investment in bonds A and B using matrix method.

[**Ans.** A = ₹ 10,000, ₹ 12000, B = ₹ 20000, ₹ 22000]

7. A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission.

Months	Sales of units			Total Commission drawn (in ₹)
	A	B	C	
January	90	100	20	800
February	130	50	40	950
March	60	100	30	850

Find out the rates of commission on items A, B and C. [**Ans.** 2%, 4% and 11%]

□□□